Representations of Toroidal general linear Superalgebra

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January 19, 2013

Abstract

We consider general linear superalgebra (type A) and tensor with Laurent polynomial ring in several variables. We then consider the universal central extension of this Lie superalgebra which we call toroidal superalgebra. We give a faithful representation of toroidal superalgebra using vertex operators and bosons.

Introduction

The purpose of this paper is to construct a faithful representation of toroidal superalgebra. To define toroidal superalgebra, let $\mathfrak{g}$ be any finite dimensional Lie superalgebra with an invariant, supersymmetric and even bilinear form. Let $A = \mathbb{C}[t_1^{\pm 1}, \cdots, t_q^{\pm 1}]$ be a Laurent polynomial ring in q commuting variables. Then $\mathfrak{g} \otimes A$ carries a natural Lie superalgebra structure. We now consider $\mathfrak{g} \otimes A \oplus \Omega_{A/d_A}$ where Ω_{A/d_A} is the module of Kahler differentials (see section 4 for details) which is central extension of $\mathfrak{g} \otimes A$ and is called

toroidal superalgebras. They were first introduce in $[EZ]$ and $[IK]$. If we take $\mathfrak{g}$ to be a Lie algebra, we get the usual toroidal Lie algebra and they are extensively studied. See for example $[EM], [EMY]$, expositary article $[E]$ and the reference there in.
In this paper we specialize to the case $\mathfrak{g} = sl(M|N)$ (See section 2 for details). In this case the toroidal superalgebra is the universal central extention of $\mathfrak{g} \otimes A$ and we denote it by τ. See $[IK]$ (We take $M > 1, N > 1$ and if $M = N$ we take $M > 2$).

In this paper we give a representations for τ using vertex operators and bosons. For a suitable non-degenerate integral lattice $\overline{\Gamma}$ and a sublattice Q, we construct a Fock space $V[\overline{\Gamma}]$. For each $\alpha \in Q$, define vertex operator $X(\alpha, z)$ such that its Fourier components $X_n(\alpha)$ act on the Fock space. We also consider the Fock space $\mathfrak{F}$ given by bosons (see $[W1]$). Then the product of vertex operators and bosons (they commute) acting on the bigger Fock space $V[\overline{\Gamma}] \otimes \mathfrak{F}$ give a representation for the toroidal superalgebra τ. It should be mentioned that, for the case $q = 1$, the representation is stated in $[KW]$ without any details whatsoever. Actually one needs a cocyle on the lattice $\overline{\Gamma}$ to get representation even in the case $q = 1$. We have worked out all the details in section 3.

In section 1 we recall the standard construction of vertex operators $[FK]$ (homogeneous picture) and state the representation for the Lie algebra case. In section 2 we give a Chevalley type basis for $gl(M|N)$ with the help of a co-cycle (2.4). In section 3, we define bosons and the Fock space $\mathfrak{F}$ on which bosons act. We now consider the bigger Fock space $V[\overline{\Gamma}] \otimes \mathfrak{F}$ and make the affine superalgebra (the toroidal superalgebra for $q = 1$) act on it (Proposition 3.3). In the last section we state and prove the main Theorem 4.6 which is to give representation to the toroidal superalgebra τ.

1 Vertex operators and Fock space

In this section we recall the basic construction $[FK]$ of vertex operators acting on Fock space. Unlike in $[FK]$ our integral lattice need not be even. The lattice contains vectors of odd norm. See the books $[K1]$ and $[Xu]$. All our vector space are over complex numbers $\mathbb{C}$.

Let d be a positive integer and let $\mathfrak{h}$ be a vector space with basis $e_1, \cdots, e_d$. Let $(,)$ be a non-degenerate bilinear form on $\mathfrak{h}$ such that $(e_i, e_j) = \delta_{ij}$. Let $\Gamma = \oplus_{i=1}^{d} \mathbb{Z}\, e_i$ be an integral lattice so that $\mathfrak{h} = \Gamma \bigotimes_{\mathbb{Z}} \mathbb{C}$.

Define a Heisenberg algebra.

$$\widehat{\mathfrak{h}} = \oplus_{k\in\mathbb{Z}} \mathfrak{h}(k) \oplus \mathbb{C}K$$

Where each $\mathfrak{h}(k)$ is an isomorphic copy of $\mathfrak{h}$ and the isomorphism given by $\alpha \longmapsto \alpha(k)$. The Lie algebra structure on $\widehat{\mathfrak{h}}$ is define by

(1.1) $$[\alpha(k), \beta(m)] = k(\alpha, \beta)\delta_{k+m,0}K$$

and K is central

Define $\mathfrak{h}_{\pm} = \oplus_{k \gtrless 0} \mathfrak{h}(k)$

The Fock space representation of $\mathfrak{h}_+ \oplus \mathfrak{h}_-$ is the symmetric algebra $S(\mathfrak{h}_-)$ of $\mathfrak{h}_-$ together with the action of $\mathfrak{h}_+ \oplus \mathfrak{h}_-$ on $S(\mathfrak{h}_-)$ defined by

K acts as 1.

$\alpha(-m)$ acts as multiplication by $\alpha(-m)$ for $m > 0$

$\alpha(m)$ acts as the unique derivation on $S(\mathfrak{h}_-)$ for which $\beta(-n) \longmapsto \delta_{m,n}\, m(\alpha, \beta)$ for $m, n > 0$

Note that $S(\mathfrak{h}_-)$ affords an irreducible representation of $\mathfrak{h}_+ \oplus \mathfrak{h}_-$.

Let $\Gamma_{\bar{i}} = \{\alpha \in \Gamma | (\alpha, \alpha) \in 2\mathbb{Z} + i\}$

Then clearly $\Gamma = \Gamma_{\bar{0}} \cup \Gamma_{\bar{1}}$

(1.2) We now define a Co-cycle

$F : \Gamma \times \Gamma \longrightarrow \{\pm 1\}$ by

$F(0, \alpha) = F(\alpha, 0) = 1$ and

$$F(e_i,e_j)=\begin{cases}1 & if\ i\le j\\ \\ -1 & if\ i>j\end{cases}$$

We extend the map bimultiplicatively to Γ.

Note that $F(e_i,-e_j)=F(e_i,e_j)$ and $F(\alpha,\beta)=F(\alpha,\beta)^{-1}$

We also note the following properties of F which can be easily checked.

$F(\alpha,\beta)F(\alpha+\beta,\gamma)=F(\beta,\gamma)F(\alpha,\beta+\gamma)$ for all $\alpha,\beta,\gamma\in\Gamma$

$F(\alpha,\beta)F(\beta,\alpha)^{-1}=(-1)^{(\alpha,\beta)+ij}$. Where $\alpha\in\Gamma_{\bar{i}}$ and $\beta\in\Gamma_{\bar{j}}$.

To see this let $\alpha=\sum m_k e_k$, $\beta=\sum n_l e_l$

Then

$$\begin{aligned}\sum m_k^2 &= (\alpha,\alpha)\equiv i(2)\\ \sum n_l^2 &= (\beta,\beta)\equiv j(2)\end{aligned}$$

and $(\alpha,\alpha)(\beta,\beta)\equiv ij(2)$

Now consider

$F(\alpha,\beta)F(\beta,\alpha)^{-1}=F(\alpha,\beta)F(\beta,\alpha)$

$=\prod\limits_{k>l}F(e_k,e_l)^{m_k n_l}\prod\limits_{l>k}F(e_l,e_k)^{n_l m_k}$

$=(-1)^D$ where $D=\sum\limits_{k\neq l}m_k n_l$

But $\sum\limits_{k\neq l}m_k n_l+\sum\limits_{k}m_k n_k=\sum\limits_{k,l}m_k n_l\equiv\sum\limits_{k,l}m_k^2 n_l^2(2)$

So

$$\begin{aligned}D &\equiv -(\alpha,\beta)+(\alpha,\alpha)(\beta,\beta)\\ &\equiv \big((\alpha,\beta)+ij\big)(2)\end{aligned}$$

It will now follow $F(\alpha,\beta)F(\beta,\alpha)^{-1}=(-1)^{(\alpha,\beta)+ij}$

(1.3) For each γ in Γ, let e^γ be a symbol and form the vector space $\mathbb{C}[\Gamma]$ with basis $\{e^\gamma,\gamma\in\Gamma\}$ over $\mathbb{C}$. We define a twisted group algebra structure on $\mathbb{C}[\Gamma]$ by

$$e^\alpha\cdot e^\beta=F(\alpha,\beta)e^{\alpha+\beta}$$

Consider the space $V[\Gamma]=\mathbb{C}[\Gamma]\otimes S(\mathfrak{h}_-)$

and define $\widehat{\mathfrak{h}}$ action on $V[\Gamma]$ by

$$\begin{aligned} \alpha(m) \cdot e^\gamma \otimes u &= e^\gamma \otimes \alpha(m)u, \; m \neq 0 \\ \alpha(0) \cdot e^\gamma \otimes u &= (\alpha, \gamma) e^\gamma \otimes u \end{aligned}$$

It is a standard fact that $V[\Gamma]$ is $\widehat{\mathfrak{h}}$ -module.

$V[\Gamma]$ has a natural $\mathbb{Z}_2$-gradation.

A vector $e^\alpha \otimes u \in V[\Gamma]$ is called even if $(\alpha, \alpha) \equiv 0(2)$ and odd if $(\alpha, \alpha) \equiv 1(2)$.

Then let $V_{\bar{o}}[\Gamma]$ be the linear span of even elements and $V_{\bar{1}}[\Gamma]$ be the linear span of odd elements. Then we have $V[\Gamma] = V_{\bar{o}}[\Gamma] \oplus V_{\bar{1}}[\Gamma]$.

An operator T on $V[\Gamma]$ is called even operator if

$$T(V_{\bar{i}}[\Gamma]) \subseteq V_{\bar{i}}[\Gamma]$$

and is called odd operator if

$$T(V_{\bar{i}}[\Gamma]) \subseteq V_{\overline{i+1}}[\Gamma]$$

Vertex operators

Let z be complex valued variable and let $\alpha \in \Gamma$.

Define $T \pm (\alpha, z) = - \sum_{n \in \mathbb{Z}_\pm} \frac{1}{n} \alpha(n) z^{-n}$

Define operator $z^{\alpha(0)}, \alpha \in \Gamma$, by $z^{\alpha(0)} \cdot e^\gamma \otimes u = z^{(\alpha,\gamma)} e^\gamma \otimes u$.

Define Vertex Operator

$Y(\alpha, z) = e^\alpha z^{\alpha(0)} \; \exp T_-(\alpha, z) \; \exp \; T_+(\alpha, z)$

For $\alpha \in \Gamma_{\bar{0}}$ define

$$X(\alpha, z) = z^{\frac{(\alpha,\alpha)}{2}} Y(\alpha, z) \text{ and}$$

write

$$X(\alpha, z) = \sum_{n \in \mathbb{Z}} X_n(\alpha) z^{-n}$$

Then $X_n(\alpha)$ is a even operator on $V[\Gamma]$.

For $\alpha \in \Gamma_{\bar{1}}$ define

$$X(\alpha, z) = Y(\alpha, z) \text{ and}$$

write

$$X(\alpha, z) = \sum_{n \in \mathbb{Z}} X_{n-\frac{1}{2}}(\alpha) z^{-n}$$

So that $X_{n-\frac{1}{2}}(\alpha)$ is an odd operator. For $\alpha \in \Gamma$, define

$$\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha(n) z^{-n-1}$$

which is an even operator.

(1.4) We will now introduce delta function and recall some standard facts from Section 2 of [FLM].

Define delta function

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n$$

Then we have the following lemma holds. See [FLM] for proof and definition. Suppose $X(z,w) = \sum_{m,n\in\mathbb{Z}} X_{m,n} z^m w^n$, define $D_z X(z,w) \ = \sum m X_{m,n} z^{m-1} w^n$

Lemma (1.5)

$$\begin{aligned}
1) X(z,w)\delta(z/w) &= X(w,w)\delta(z/w) \\
2) X(z,w)\delta(z,w) &= X(z,z)\delta(z/w) \\
3) X(z,w)D_z(\delta(z/w)) &= X(w,w)D_z(\delta(z/w)) - (D_z X)(w,w)\delta(z/w) \\
4) X(z,w)D_w(\delta(z/w)) &= X(z,z)D_w(\delta(z/w)) - (D_w X)(z,z)\delta(z/w)
\end{aligned}$$

Let $\Delta = \{\alpha_{ij} = e_i - e_j, i \neq j\} \subseteq \Gamma$

Then Δ is a finite root system of type A. The following is a well known result. See [FK] or [EM].

Proposition (1.6) The operator $\alpha_{ij}(m), X_n(\alpha_{ij}),\ K = Id$ for $i \neq j, m, n \in \mathbb{Z}$, defines a representation of an affine Lie-algebra of type A on $V[\Gamma]$.

(1.7) We will now introduce normal ordering and super commutator on operators $Y(\alpha, z)$.

Suppose $\alpha \in \Gamma_{\bar{i}}$ and $\beta \in \Gamma_{\bar{j}}$

$$: Y_n(\alpha) Y_m(\beta) := \begin{cases} Y_n(\alpha)Y_m(\beta) & \text{if } n \leq m \\ (-1)^{ij} Y_m(\beta) Y_n(\alpha) & \text{if } n > m \end{cases}$$

$$[Y_n(\alpha), Y_m(\beta)] = Y_n(\alpha)Y_m(\beta) - (-1)^{ij} Y_m(\beta) Y_n(\alpha)$$

We now prove the following

Lemma (1.8): Let $\alpha \in \Gamma_{\bar{i}}, \beta \in \Gamma_{\bar{j}}$

1) $[Y(\alpha, z), Y(\beta, w) = 0 \; if \; (\alpha, \beta) \geq 0$

2) $[Y(\alpha, z), Y(\beta, w)] = F(\alpha, \beta)Y(\alpha + \beta, z)z^{-1}\delta(z/w)$ if $(\alpha, \beta) = -1$

3) $[\alpha(z), Y(\beta, w)] = (\alpha, \beta)z^{-1}Y(\beta, z)\delta(z/w)$

Proof We have $z^{\alpha(0)}e^{\beta} = z^{(\alpha,\beta)}e^{\beta}z^{\alpha(0)}$

Note that $[T_{-}(\alpha, z), T_{+}(\beta, w)] = \log(1 - w/z)^{(\alpha,\beta)}$ which will follow from (6.1.57)) of [Xu].

Let $T(\alpha, \beta, z, w) = e^{\alpha+\beta}z^{\alpha(0)}w^{\beta(0)} \; exp\, T_{-}(\alpha, z) exp\, T_{-}(\beta, w) \; exp\, T_{+}(\alpha, z) \; exp\, T_{+}(\beta, w)$

Consider

$e^{\alpha}z^{\alpha(0)} \; exp \; T_{-}(\alpha, z) \; exp \; T_{+}(\alpha, z).\; e^{\beta}z^{\beta(0)} \; exp \; T_{-}(\beta, w) \; exp \; T_{+}(\beta, w)$

$= F(\alpha, \beta)T(\alpha, \beta, z, w)z^{(\alpha,\beta)}(1 - w/z)^{(\alpha,\beta)}$

By symmetry we have

$$Y(\alpha, z)Y(\beta, w) - (-1)^{ij}Y(\beta, w)Y(\alpha, z)$$

$$= T(\alpha, \beta, z, w)(F(\alpha, \beta)z^{(\alpha,\beta)}(1 - w/z)^{(\alpha,\beta)}$$

$$-(-1)^{ij}F(\beta, \alpha)w^{(\beta,\alpha)}(1 - z/w)^{(\alpha,\beta)})$$

Consider the term in brackets which is equal to

$$F(\alpha, \beta)\left(z^{(\alpha,\beta)}(1 - w/z)^{(\alpha,\beta)} - (-1)^{ij}F(\alpha, \beta)^{-1}F(\beta, \alpha)w^{(\alpha,\beta)}(1 - z/w)^{(\alpha,\beta)}\right)$$

From (1.2) we have

$$(-1)^{ij}F(\alpha, \beta)^{-1}F(\beta, \alpha) = (-1)^{(\alpha,\beta)}$$

So the term in the brackets equal to

$$F(\alpha, \beta)\left((z - w)^{(\alpha,\beta)} - (-1)^{(\alpha,\beta)}(w - z)^{(\alpha,\beta)}\right)$$

which is equal to zero if $(\alpha, \beta) \geq 0$. Thus we have proved (1) of the Lemma.

Suppose $(\alpha, \beta) = -1$. Now the term in the bracket equal to

$$F(\alpha, \beta)\left(z^{-1}(1 - w/z)^{-1} + w^{-1}(1 - z/w)^{-1}\right)$$

$$= w^{-1}F(\alpha, \beta)\left(\tfrac{w}{z}(1 - w/z)^{-1} + (1 - z/w)^{-1}\right)$$

$$= F(\alpha, \beta)w^{-1}\delta(z/w)$$

Then we have

$$[Y(\alpha, z), Y(\beta, w)] = F(\alpha, \beta)T(\alpha, \beta, z, w).w^{-1}\delta(z/w)$$

Now by Lemma 1.5(2)

$$\begin{aligned} &= F(\alpha, \beta)T(\alpha, \beta, z, z)z^{-1}\delta(z/w) \\ &= F(\alpha, \beta)Y(\alpha + \beta, z)z^{-1}\delta(z/w) \end{aligned}$$

This proves the second part of the Lemma. For the third part of the lemma we refer to (6.1.56) of $[Xu]$

We need the following

Corollary (1.9)

(1) For $j \neq k,\ [X_{m-\frac{1}{2}}(e_i), X_n(e_j - e_k)] = \delta_{ik}F(e_i, e_j - e_k)X_{m+n-\frac{1}{2}}(e_j)$

(2) $[X_{m-\frac{1}{2}}(e_i), X_{n+\frac{1}{2}}(-e_j)] = \delta_{ij}F(e_i, -e_j)\delta_{m+n,o}K.$

(3) Suppose $\alpha, \beta \in \Gamma,\ [\alpha(m), X_n(\beta)] = (\alpha, \beta)X_{m+n}(\beta)$

We now recall the following known facts. Recall $\alpha_{ij} = e_i - e_j$

(**1.10**) (1) $[X_m(\alpha_{ij}), X_n(-\alpha_{ij})] = F(\alpha_{ij}, -\alpha_{ij})(\alpha_{ij}(m + n) + \delta_{m+n,0}\ mK)$

2) For $i \neq j$

$$\sum_{k \in \mathbb{Z}} : X_{k+\frac{1}{2}}(e_i)X_{n-k-\frac{1}{2}}(-e_j) := F(e_i, -e_j)X_n(e_i - e_j)$$

$$\sum_{k \in \mathbb{Z}} : X_{n-k-\frac{1}{2}}(-e_j)X_{k+\frac{1}{2}}(e_i) := F(-e_j, e_i)X_n(e_i - e_j)$$

3): $X(e_i, z)X(-e_i, z) := e_i(z)$

In other words

$\sum_{k \in \mathbb{Z}} : X_{k+\frac{1}{2}}(e_i)X_{n-k-\frac{1}{2}}(-e_i) := e_i(n)$

1.10(1) is very standard. For example see $[KF]$ or $[EM]$.

1.10(2) Follows from the definition of vertex operators.

Note that equations are equivalent as the operator at LHS anti-commute and $F(e_i, -e_j) = -F(-e_j, e_i)$.

1.10(3) Which is non-trivial can be found in (5.37) of $[FZ]$

Proposition (1.11)

The operators $e_k(n), X_m(\alpha_{ij}), 1 \leq i,j,k \leq d,\ n,m \in \mathbb{Z},\ K = Id$ defines a representation of $\overset{\wedge}{gl_d}$ on the space $V[\Gamma]$. Where $\overset{\wedge}{gl_d}$ is the standard affinization of gl_d.

Proof In view of Proposition 1.6, the only thing we need to check is

$$[e_k(r), X_m(\alpha_{ij})] = (e_k, \alpha_{ij})X_{m+r}(\alpha_{ij})$$

This will follow from Corollary 1.9(3).

2 Affine Superalgebras

(2.1) A lie superalgebra is a $\mathbb{Z}_2-$ graded vector space $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ equipped with $\mathbb{C}$ bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie superbracket, satisfying the conditions.

(1) $[\mathfrak{g}_{\overline{i}}, \mathfrak{g}_{\overline{j}}] \subseteq\ \mathfrak{g}_{\overline{i+j}}$

(2) $[X,Y] = -(-1)^{ij}[Y,X]$

(3) $\big[[X,Y],Z\big] = \big[X,[Y,Z]\big] - (-1)^{ij}\big[Y,[X,Z]\big]$

For all homogeneous elements $X \in \mathfrak{g}_{\overline{i}}, Y \in \mathfrak{g}_{\overline{j}}$ and $Z \in \mathfrak{g}_{\overline{k}}$

The subspace $\mathfrak{g}_{\overline{o}}$ is called even and the subspace $\mathfrak{g}_{\overline{1}}$ is called odd. It is easy to see that $\mathfrak{g}_{\overline{o}}$ is the usual Lie algebra and $\mathfrak{g}_{\overline{1}}$ is $\mathfrak{g}_{\overline{o}}$ module. The identity (3) is called super Jacobi identity.

(2.2) An important example of a Lie superalgebra is the space of all endomorphisms of a $\mathbb{Z}_2-$ graded vector space $V = V_{\overline{o}} \oplus V_{\overline{1}}$. We assume $\dim V_{\overline{o}} = M$ and $\dim V_{\overline{1}} = N$. The $\mathbb{Z}_2-$ gradation on V naturally induces a $\mathbb{Z}_2-$ gradation on

$$End(V) = (End(V))_{\overline{o}} \oplus (End(V))_{\overline{1}}$$

by letting

$$End(V)_j = \{f \in End(V) : f(V_k) \subseteq V_{k+j} \text{for all } k \in \mathbb{Z}_2\}$$

End V becomes a Lie superalgebra with Lie super bracket

$$[f,g] = f \circ g - (-1)^{ij} g \circ f$$

for all $f \in End(V)_i, g \in End(V)_j$. We fix a basis of $V_{\overline{o}}$ say $e_1, \cdots e_M$ and a basis $e_{M+1}, \cdots e_{M+N}$ of $V_{\overline{1}}$. We denote the Lie superalgebra $End(V)$ by $gl(M|N)$ which is of type A.

An elements f in $End(V)$ can be represented by a matrix form with respect to the above basis.

$$f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is $M \times M$ matrix, B is $N \times M$ matrix, C is $M \times N$ matrix and D is $N \times N$ matrix.

It is easy to see that

$$\begin{pmatrix} A & O \\ O & D \end{pmatrix} \in End(V)_{\overline{o}} \; and$$

$$\begin{pmatrix} O & B \\ C & O \end{pmatrix} \in End(V)_{\overline{1}}$$

If $X \in End(V) = gl(M|N)$ is a homogeneous of deg i, then we denote $|X| = i$.

(2.3) Define super trace of $f, strf = Trace\ A - Trace\ D$.

Let $<,>$ be a form on $EndV$ by $< f, g >= str(f \circ g)$.

The form $<,>$ satisfy by the following properties

(1) $< X, Y >= (-1)^{|X||Y|} < Y, X >$ (Super-symmetric)

(2) $< [X,Y], Z >=< X, [Y,Z] >$ (invariant)

(3) $< gl(M|N)_{\overline{o}} \mid \mathfrak{g}l(M|N)_{\overline{1}} >= 0$ (even)

The root system of $gl(M|N)$ is

$$\Delta = \{\alpha_{ij} = e_i - e_j, i \neq j, 1 \leq i,j \leq M+N\}$$

(2.4) We will now give a Chevalley type basis with the help of a Co-cycle for $gl(M|N)$.

Let T be a vector space spanned by vector $T_{ij}, 1 \leq i,j \leq M+N$ We call T_{ij} even if $1 \leq i,j \leq M$ or $M+1 \leq i,j \leq M+N$. We call T_{ij} odd if $1 \leq i \leq M$ and $j > M$ or $1 \leq j \leq M$ and $i > M$. Let T_0 be the linear span of even elements and T_1 be the linear span of odd elements. Then we have $T = T_0 \oplus T_1$ is a $\mathbb{Z}_2-$ graded space. We would like to give Lie superalgebra structure on T.

Let $\Gamma = \oplus_{i=1}^{M} \mathbb{Z} e_i$ and let

$F : \Gamma \times \Gamma \to \{\pm 1\}$ be the Co-cycle as defined in (1.2).

Let $\alpha_{ij} = e_i - e_j$. Note that $(\alpha_{ij}, \alpha_{kl}) \geq -2$.

We will now define a super bracket on T.

$(T1)$ $1 \leq i,j,k,l \leq M$

$i \neq j, k \neq l$

$$[T_{ij}, T_{kl}] = \begin{cases} 0 \text{ if } (\alpha_{ij}, \alpha_{kl}) \geq 0 \\ F(\alpha_{ij}, \alpha_{kl}) T_{il} \text{ if } j = k, l \neq i, (\alpha_{ij}, \alpha_{kl}) = -1 \\ F(\alpha_{ij}, \alpha_{kl}) T_{kj} \text{ if } l = i, j \neq k, (\alpha_{ij}, \alpha_{kl}) = -1 \\ F(\alpha_{ij}, \alpha_{kl}) (T_{ii} - T_{jj}) \text{ if } l = i, j = k, (\alpha_{ij}, \alpha_{kl}) = -2 \end{cases}$$

$i = j, k \neq l$

$[T_{ii}, T_{kl}] = (e_i, \alpha_{kl}) T_{kl}$

$i = j, k = l$

$[T_{ii}, T_{kk}] = 0$

$(T2)$ $1 \leq i,j,k,l \leq N$

$[T_{i+M,j+M}, T_{k+M,l+M}] = \delta_{jk} T_{i+M,l+M} - \delta_{il}\ T_{k+M,j+M}$

$(T3)$ $1 \leq i \neq j, l \leq M, 1 \leq k \leq N$

$[T_{ij}, T_{k+M,l}] = \delta_{il} F(e_j, e_i) T_{k+M,j}$

$[T_{k+M,l} T_{ij}] = \delta_{il} F(e_i, e_j) T_{k+M,j}$

$[T_{ii}, T_{k+M,l}] = -\delta_{il} T_{k+M,l}$

$[T_{k+M,l}, T_{ii}] = \delta_{il} T_{k+M,l}$

$(T4)$ $1 \leq i,j,k \leq M, 1 \leq l \leq N$

$[T_{ij}, T_{k,l+M}] = \delta_{jk} F(e_i, e_j) T_{i,l+M}$

$[T_{k,l+M}, T_{ij}] = -\delta_{jk} F(e_i, e_j) T_{i,l+M}$

$(T5)$ $1 \le i, j, k \le N, 1 \le l \le M$

$[T_{i+M,j+M}, T_{k+M,l}] = \delta_{jk} T_{i+M,l}$

$[T_{k+M,l}, T_{i+M,j+M}] = -\delta_{jk} T_{i+M,l}$

$(T6)$ $1 \le i, j, l \le N, 1 \le k \le M$

$[T_{i+M,j+M}, T_{k,l+M}] = -\delta_{li} T_{k,j+M}$

$[T_{k,l+M}, T_{i+M,j+M}] = \delta_{li} T_{k,j+M}$

$(T7)$ $1 \le i, l \le N, 1 \le j, k \le M$

$[T_{i+M,j}, T_{k,l+M}] = \delta_{jk} T_{i+M,l+M} + F(e_k, e_j) \delta_{li} T_{kj}$

$[T_{k,l+M}, T_{i+M,j}] = \delta_{jk} T_{i+M,l+M} + F(e_k, e_j) \delta_{li} T_{kj}$

$(T8)$ $1 \le i, j \le M, 1 \le k, l \le N$

$[T_{ij}, T_{k+M,l+M}] = [T_{k+M,l+M}, T_{ij}] = 0$

$(T9)$ $1 \le i, k \le N, 1 \le j, l \le M$

$[T_{i+M,j}, T_{k+M,l}] = [T_{k+M,l}, T_{i+M,j}] = 0$

$(T10)$ $1 \le i, k \le M, 1 \le j, l \le N$

$[T_{i,j+M}, T_{k,l+M}] = [T_{k,l+M}, T_{i,j+M}] = 0$

We claim that T is a Lie superalgebra with the above super bracket. It is sufficient to check the super Jacobi identity. This can be directly checked case by case but very tedious.

Notice that the super bracket is antisymmetric for vectors of type even-even or even-odd. The super bracket is symmetric for odd-odd vectors. This is consistent with Lie superalgebra definition.

Let $\mathfrak{h}$ be the subspace spanned by $T_{ii}, 1 \le i \le M+N$. Clearly $\mathfrak{h}$ is abelian and T decomposes with respect to $\mathfrak{h}$.
So by comparing the root system of T with that of $gl(M|N)$ we conclude that $T \cong gl(M|N)$. See $[K2]$ for details.

(2.5) We now define bilinear form $(,)_T$ on T

(1) $1 \le i, j, k, l \le M$

$(T_{ij}, T_{kl})_T = F(\alpha_{ij}, \alpha_{kl}) \delta_{jk} \delta_{li}$

(2) $1 \leq j, k \leq M, 1 \leq i, l \leq N$

$(T_{i+M,j}, T_{k,l+M})_T = -\delta_{jk}\delta_{li}$

$(T_{k,l+M}, T_{i+M,j})_T = \delta_{jk}\delta_{li}$

(3) $1 \leq i, j, k, l \leq N$

$(T_{i+M,j+M}, T_{k+M,l+M})_T = -\delta_{jk}\delta_{li}$

(4) All other brackets are zero.

Extend$(,)_T$ to whole of T bilinearly. It is straight forward to check that $(,)_T$ is a supersymmetric,invariant and even.

We will now define affine superalgebra

$$\widehat{T} = T \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

Let $X(m) = X \otimes t^m, X \in T, m \in \mathbb{Z}$.

$[X(m), Y(n)] = [X, Y](m + n) + (X, Y)_T \ \delta_{m+n,o} \ mK$

We will now write down the affine bracket explicitly on the genarators of T.

$(R1)$ $1 \leq i, j, k, l \leq M$

$i \neq j, k \neq l$

$$[T_{ij}(m), T_{kl}(n)] = \begin{cases} 0 \text{ if } (\alpha_{ij}, \alpha_{kl}) \geq 0 \\ F(\alpha_{ij}, \alpha_{kl})T_{il}(m+n) \text{ if } j = k, l \neq i, (\alpha_{ij}, \alpha_{kl}) = -1 \\ F(\alpha_{ij}, \alpha_{kl})T_{kj}(m+n) \text{ if } l = i, j \neq k, (\alpha_{ij}, \alpha_{kl}) = -1 \\ F(\alpha_{ij}, \alpha_{kl})((T_{ii} - T_{jj})(m+n) + m\delta_{m+n,0}K) \text{if } l = i, j = k, (\alpha_{ij}, \alpha_{kl}) = -2 \end{cases}$$

$i = j, k \neq l$

$[T_{ii}(m), T_{kl}(n)] = (e_i, \alpha_{kl})T_{kl}(m + n)$

$i = j, k = l$

$[T_{ii}(m), T_{kk}(n)] = m \ \delta_{ik} \ \delta_{m+n,0}K$

$(R2)$ $1 \leq i, j, k, l \leq N$

$[T_{i+M,j+M}(m), T_{k+M,l+M}(n)] =$

$\delta_{jk}T_{i+M,l+M}(m + n) \ - \delta_{li}T_{k+M,l+M}(m + n) \ - \delta_{jk}\delta_{il} \ m \ \delta_{m+n,o}K$

$(R3)$ $1 \leq i \neq j, l \leq M, 1 \leq k \leq N$

$[T_{ij}(m), T_{k+M,l}(n)] = \delta_{il}F(e_j, e_i)T_{k+M,j}(m + n)$

$[T_{k+M,l}(n), T_{ij}(m)] = \delta_{il}F(e_i, e_j)T_{k+M,j}(m + n)$

$[T_{ii}(m), T_{k+M,l}(n)] = -\delta_{il} T_{k+M,l}(m+n)$

$[T_{k+M,l}(n), T_{ii}(m)] = \delta_{il} T_{k+M,l}(m+n)$

$(R4)\ 1 \leq i, j, k \leq M, 1 \leq l \leq N$

$[T_{ij}(m), T_{k,l+M}(n)] = \delta_{jk} F(e_i, e_j) T_{i,l+M}(m+n)$

$[T_{k,l+M}(n), T_{ij}(m)] = -\delta_{jk} F(e_i, e_j) T_{i,l+M}(m+n)$

$(R5)\ 1 \leq i, j, k \leq N, 1 \leq l \leq M$

$[T_{i+M,j+M}(m), T_{k+M,l}(n) = \delta_{jk} T_{i+M,l}(m+n)$

$[T_{k+M,l}(n), T_{i+M,j+M}(m)] = -\delta_{jk} T_{i+M,l}(m+n)$

$(R6)\ 1 \leq i, j, l \leq N, 1 \leq k \leq M$

$[T_{i+M,j+M}(m), T_{k,l+M}(n) = -\delta_{li} T_{k,j+M}(m+n)$

$[T_{k,l+M}(n), T_{i+M,j+M}(m)] = \delta_{li} T_{k,j+M}(m+n)$

$(R7)\ 1 \leq i, l \leq N, 1 \leq j, k \leq M$

$[T_{i+M,j}(m), T_{k,l+M}(n)] =$

$\delta_{jk} T_{i+M,l+M}(m+n) + F(e_k, e_j) \delta_{li}\ T_{kj}(m+n)$

$-\delta_{jk}\ \delta_{li}\ m\ \delta_{m+n,o}\ K$

$[T_{k,l+M}(n), T_{i+M,j}(m)] =$

$\delta_{jk}\ T_{i+M,l+M}\ (m+n) + F(e_k, e_j)\ \delta_{li}\ T_{kj}\ (m+n)$

$+\delta_{jk}\ \delta_{li}\ n\ \ \delta_{m+n,o}\ K.$

$(R8)\ 1 \leq i, j \leq M, 1 \leq K, l \leq N$

$[T_{ij}(m), T_{k+M,l+M}(n)] = [T_{k+M,l+M}(n), T_{ij}(m)] = o$

$(R9)\ 1 \leq i, k \leq N, 1 \leq j, l \leq M$

$[T_{i+M,j}(m), T_{k+M,l}(n)] = [T_{k+M,l}(n), T_{i+M,j}(m)] = o$

$(R10)\ 1 \leq i, k \leq M, 1 \leq j, l \leq N$

$[T_{i,j+M}(m), T_{k,l+M}(n)] = [T_{k,l+M}(n), T_{i,j+M}(m)] = o$

(2.6) In this subsection we will state some identities on operators acting on a superalgebra.

A $\mathbb{Z}_2$-graded algebra is called super algebra. Let $A = A_0 \oplus A_1$ be $\mathbb{Z}_2$- graded algebra. The elements of A_0 are called even and the elements of A_1 are called

odd. An operator on A is called even operator if it take even space to even space and odd space to odd. An operator is called odd if it takes even space to odd and takes odd space to even. An operator is called homogeneous if it is even or odd. Suppose X is homogeneous operator then let

$$\begin{aligned} \mid X \mid = 0 \quad if\ X\ is\ even \\ \mid X \mid = 1 \quad if\ \ X\ is\ odd \end{aligned}$$

Suppose X and Y are homogeneous operator on A.
Define $[X,Y] = XY - (-1)^{|X|\ |Y|} YX$
Note that $[Y,X] = -(-1)^{|X|\ |Y|}[X,Y]$.

The following can easily be verified. For example see [K2].
Suppose X, Y, Z are homogeneous operators on A. Then

(2.7) (1)$[X, YZ] = [X,Y]Z + (-1)^{|X|\ |Y|} Y[X,Z]$
(2) $[XY, Z] = X[Y,Z] + (-1)^{|Z|\ |Y|}[X,Z]Y$

We need the following Lemma.

Lemma 2.8 Suppose X_1 and X_2 are odd operators and Y_1 and Y_2 are even operators. Further suppose $[X_i, Y_j] = 0$ for all i and j. Then

$$[X_1Y_1, X_2Y_2] = [X_1, X_2]Y_1Y_2 - X_2X_1[Y_1, Y_2].$$

Proof Director verification. Just expand the brackets. Note that X_1Y_1 and X_2Y_2 and odd operators.

3 Bosons and Fockspace

In this section we recall the definitions of bosonic fields acting on a Fock space $\mathfrak{F}$ from [W2].
For $1 \leq j \leq N$ let

$$\begin{aligned} \varphi^j(z) &= \textstyle\sum_{r\in\mathbb{Z}} \varphi^j_{r-\frac{1}{2}} z^{-r}\ and \\ \varphi^{j^*}(z) &= \textstyle\sum_{r\in\mathbb{Z}} \varphi^{j^*}_{r-\frac{1}{2}} z^{-r} \end{aligned}$$

$2N$ bosonic fields and the coefficients $\varphi^j_{r-\frac{1}{2}}, \varphi^{j^*}_{r-\frac{1}{2}}$ acting on a Fock space $\mathfrak{F}$ satisfying

(3.1) (1) $\varphi^i_{r-\frac{1}{2}}\varphi^j_{s-\frac{1}{2}} - \varphi^j_{s-\frac{1}{2}}\varphi^i_{r-\frac{1}{2}} = 0$

(2) $\varphi^{i*}_{r-\frac{1}{2}}\varphi^{j*}_{s-\frac{1}{2}} - \varphi^{j*}_{s-\frac{1}{2}}\varphi^{i*}_{r-\frac{1}{2}} = 0$

(3) $\varphi^i_{r-\frac{1}{2}}\varphi^{j*}_{s-\frac{1}{2}} - \varphi^{j*}_{s-\frac{1}{2}}\varphi^i_{r-\frac{1}{2}} = -\delta_{r+s-1,0}\delta_{ij}$

Then the fock space $\mathfrak{F}$ admits a vector $|0>$, called the vacuum with the following properties.

$$\varphi^i_{r-\frac{1}{2}}|0>=\varphi^{j^*}_{s-\frac{1}{2}}|0>=0\ \forall i,\ j, \forall\ r,s>0$$

Further $\varphi^i_{r-\frac{1}{2}}, \varphi^{j*}_{s-\frac{1}{2}},\ r,s\leq 0$ act freely on $\mathfrak{F}$ and any element of the Fock space $\mathfrak{F}$ is linear combinations of elements obtained by applying

$$\varphi^i_{r-\frac{1}{2}}, \varphi^{j*}_{s-\frac{1}{2}}, r,s\leq 0$$

The operators $\varphi^i_{r-\frac{1}{2}}, \varphi^{j*}_{s-\frac{1}{2}}, r,s>0$ (respectively $r,s\leq 0$) are called annihilation (respectively creation) operators.

The purpose of this section is to give representation of $\overset{\wedge}{gl}\ (M/N)$ on the space $V[\Gamma]\otimes\mathfrak{F}$. The vertex operators $X_n(\alpha), \alpha(n)$ acting on the first component and the bosons $\varphi^i_{r-\frac{1}{2}}$ and $\varphi^{j*}_{s-\frac{1}{2}}$ act on the second component. In particular the vertex operators and bosons commute.

Note that $V[\Gamma]\otimes\mathfrak{F}$ is a $\mathbb{Z}_2$ -graded space with $e^\alpha\otimes u\otimes v$ is even if (α,α) is even and odd if (α,α) is odd.

(3.2) We now give a representation of $\hat{T}$ on the space $V[\Gamma]\otimes\mathfrak{F}$ by the following rule

(1) $1\leq i\leq M,$

$T_{ii}(z)\mapsto e_i(z) =: X(e_i,z)X(-e_i,z):.$

(2) $i\neq j,\ 1\leq i,j\leq M$

$T_{ij}(z)\mapsto X(e_i-e_j,z)$

(3) $1\leq i\leq M,\ 1\leq j\leq N$

$T_{i,j+M}(z)\mapsto X(e_i,z)\varphi^{j^*}(z)$

(4) $1\leq j\leq N,\ 1\leq i\leq M$

$T_{j+M,i}(z)\mapsto X(-e_i,z)\varphi^j(z)$

(5) $1\leq i,\ j\leq N$

$T_{i+M,j+M}(z)\mapsto: \varphi^i(z)\varphi^{j^*}(z):$

(6) $K\mapsto Id.$

Here normal ordering is the following

$$
\begin{aligned}
:\varphi^{i}_{r-\frac{1}{2}}\varphi^{j^*}_{s-\frac{1}{2}}: \quad &= \varphi^{i}_{r-\frac{1}{2}}\varphi^{j^*}_{s-\frac{1}{2}} \ if \ r \leq s \\
&= \varphi^{j^*}_{s-\frac{1}{2}}\varphi^{i}_{r-\frac{1}{2}} \ if \ r > s
\end{aligned}
$$

Note that normal ordering is not necessary in (3) and (4) as the operators commute. Normal ordering for (1) is defined in the earlier section.

We will now state the main result of this section. It is stated in [KW] but without any proof. We could not find the details anywhere in the literature. So we decided to offer the details of the proof. It should be mentioned that we need a Co-cycle and this is not given in $[KW]$.

For $1 \leq i \leq M, \ 1 \leq j \leq N$

Write

$$S_{i,j+M}(z) = X(e_i, z)\varphi^{j^*}(z) = \sum_{m\in\mathbb{Z}} S_{i,j+M}(m) z^{-m-1}$$

So that

$$\sum_{r\in\mathbb{Z}} X_{r-\frac{1}{2}}(e_i)\varphi^{j^*}_{m-r+\frac{1}{2}} = S_{i,j+M}(m)$$

For $1 \leq j \leq N, \ 1 \leq i \leq M$

Write

$$S_{j+M,i}(z) = X(-e_i, z)\varphi^{j}(z) = \sum_{m\in\mathbb{Z}} S_{j+M,i}(m) z^{-m-1}$$

So that

$$\sum_{r\in\mathbb{Z}} X_{r-\frac{1}{2}}(-e_i)\varphi^{j}_{m-r+\frac{1}{2}} = S_{j+M,i}(m)$$

For $1 \leq i, \ j \leq N$

Write

$$S_{i+M,j+M}(z) :=: \varphi^{i}(z)\varphi^{j^*}(z) := \sum_{m\in\mathbb{Z}} S_{i+M,j+M}(m) z^{-m-1}$$

So that

$$\sum_{r\in\mathbb{Z}} :\varphi^{i}_{r-\frac{1}{2}}\varphi^{j^*}_{m-r+\frac{1}{2}} := S_{i+M,j+M}(m)$$

Proposition 3.3 Notation as above. The above map defines a representation of $\hat{T}\cong\overset{\wedge}{gl}\ (M/N)$ on the space $V[\Gamma]\otimes\mathfrak{F}$.

Proof The only thing that we need to check is relations $(R1)$ to $(R10)$ in **(2.5)**.

Let $\mathbb{R}$ be the field of real numbers.

Let

$$\begin{aligned} \psi : \mathbb{R} &\rightarrow \{0,1\}\ such\ that \\ \psi(x) &= 1\ if\ \mid x \mid \leq 1 \\ &= 0\ if\ \mid x \mid > 1 \end{aligned}$$

We use the principle of truncation to deal with infinite sums. Consider $S_{ij}(m)$ which is infinite sum but $S_{ij}(m)\psi(\epsilon m)$ is finite and tends to $S_{ij}(m)$ as $\epsilon \rightarrow 0$. The relation is (R1) is standard which is nothing but standard vertex operator construction (homogeneous picture). For example see [FK] or [EM]. We will first verify (R7) which is most important relation. For that it is sufficient to check the following:

$$\begin{aligned} 3.3(1)\quad & [S_{i+M,j}(m), S_{k,l+M}(n)] \\ & = \delta_{jk} S_{i+M,l+M}(m+n) + F(e_k, e_j)\delta_{li} X_{m+n}(e_k - e_j) \\ & -\delta_{jk}\delta_{li} \cdot m\delta_{m+n,0}, \text{if } j \neq k \\ & = \delta_{jk} S_{i+M,l+M}(m+n) + \delta_{li}\ e_k(m+n) \\ & -\delta_{jk}\ \delta_{li}\ m\ \delta_{m+n,0}\ if\ j = k \end{aligned}$$

$$\begin{aligned} 3.3(2)\quad & [S_{k,l+M}\ (n),\ S_{i+M,j}(m)] \\ & = \delta_{jk} S_{i+M,l+M}(m+n) + F(e_k, e_j)\delta_{li} X_{m+n}(e_k - e_j) \\ & +\delta_{jk}\delta_{li}\ n\ \delta_{m+n,0}\ if\ j \neq k \\ & = \delta_{jk} S_{i+M,l+M}(m+n) + \delta_{li}\ e_k(m+n) \\ & +\delta_{jk}\delta_{li}\ n\ \delta_{m+n,0}\ if\ j = k \end{aligned}$$

We will first verify (R7) which is most important.

First consider

$$A = [S_{i+M,j}(m), S_{k,l+M}(n)]\ \psi(\epsilon n) = \sum_r \sum_s \left[X_{r-\frac{1}{2}}(-e_j)\varphi^i_{m-r+\frac{1}{2}}, X_{s-\frac{1}{2}}(e_k)\varphi^{l^*}_{n-s+\frac{1}{2}} \right] \psi(\epsilon n)$$

Recall that X operator are odd and φ operators are even and X operators commute with φ operators. So we can use Lemma (2.8). So

$$A = \sum_{r,s \in \mathbb{Z}} \left[X_{r-\frac{1}{2}}(-e_j), X_{s-\frac{1}{2}}(e_k) \right] \varphi^i_{m-r+\frac{1}{2}} \varphi^{l^*}_{n-s+\frac{1}{2}} \psi(\epsilon n)$$

$$-\sum_{r,s\in\mathbb{Z}} X_{s-\frac{1}{2}}(e_k)X_{r-\frac{1}{2}}(-e_j)\left[\varphi^i_{m-r+\frac{1}{2}},\varphi^{l^*}_{n-s+\frac{1}{2}}\right]\psi(\epsilon n)$$

We will now use 3.1(3) and Corollary 1.9(2) to conclude

$$\begin{aligned} A &= \delta_{jk}\textstyle\sum_{r,s\in\mathbb{Z}} F(e_k,-e_j)\delta_{r+s-1,0}\ \varphi^i_{m-r+\frac{1}{2}}\varphi^{l^*}_{n-s+\frac{1}{2}}\psi(\epsilon n)\\ &+\delta_{il}\textstyle\sum_{r,s\in\mathbb{Z}} X_{s-\frac{1}{2}}(e_k)X_{r-\frac{1}{2}}(-e_j)\delta_{m+n-(r+s)+1,0}\psi(\epsilon n)\\ &= \delta_{jk}F(e_k,-e_j)\textstyle\sum_{s\in\mathbb{Z}}\ \varphi^i_{m+s-\frac{1}{2}}\ \varphi^{l^*}_{n-s+\frac{1}{2}}\psi(\epsilon n)\\ &+\delta_{il}\textstyle\sum_{s\in\mathbb{Z}} X_{s-\frac{1}{2}}(e_k)X_{m+n-s+\frac{1}{2}}(-e_j)\psi(\epsilon n)\end{aligned}$$

We will now normal order the sums. For this we split the first term according to $m+s-\frac{1}{2}\le n-s+\frac{1}{2}$ (which is in the normal ordering) or $m+s-\frac{1}{2}>n-s+\frac{1}{2}$. Similarly we split the second sum according to $s-\frac{1}{2}\le m+n-s+\frac{1}{2}$ or $s-\frac{1}{2}>m+n-s+\frac{1}{2}$

Thus

$$\begin{aligned} A &= \delta_{jk}\sum_{s\epsilon\mathbb{Z}}F(e_k,-e_j):\varphi_{m+s-\frac{1}{2}}\ \varphi^{l^*}_{n-s+\frac{1}{2}}:\psi(\epsilon n)\\ &- \delta_{jk}\delta_{il}\textstyle\sum_{2s>n-m+1}F(e_k,-e_j)\psi(\epsilon n)\delta_{m+n,0}\\ &+ \delta_{il}\sum_{s\epsilon\mathbb{Z}}:X_{s-\frac{1}{2}}(e_k)X_{m+n-s+\frac{1}{2}}(-e_j):\psi(\epsilon n)\\ &+ \delta_{il}\ \delta_{jk}\ \textstyle\sum_{2s>m+m+1}\psi(\epsilon n)\delta_{m+n,0}\end{aligned}$$

$$\begin{aligned} A &= \delta_{jk}S_{i+M,l+M}\psi(\epsilon n)\\ &- \delta_{jk}\delta_{il}\textstyle\sum_{s>n+\frac{1}{2}}\psi(\epsilon n)\delta_{m+n,0}\\ &+ \delta_{il}\sum_{s\in\mathbb{Z}}:X_{s-\frac{1}{2}}(e_k)\ X_{m+n-s+\frac{1}{2}}(-e_j):\psi(\epsilon n)\\ &+ \delta_{il}\delta_{jk}\textstyle\sum_{s>\frac{1}{2}}\psi(\epsilon n)\delta_{m+n,0}\end{aligned}$$

Now it is easy to see that

$$\begin{aligned}&\textstyle\sum_{s>\frac{1}{2}}\psi(\epsilon n)\delta_{m+n,0}-\sum_{s>n+\frac{1}{2}}\psi(\epsilon n)\delta_{m+n,0}\\ &= n\delta_{m+n,0}\psi(\epsilon n)\end{aligned}$$

Now taking $\epsilon\to 0$ we see that for $j\neq k$ (Using Corollary 1.9(1))

$$A\to\delta_{jk}S_{i+M,l+M}(m+n)+\delta_{il}F(e_k,-e_j)X_{m+n}(e_k-e_j)+\delta_{jk}\ \delta_{li}\ n\delta_{m+n,0}$$

Suppose $j=k$ (Using 1.10(3))

$$\begin{aligned} A &= \delta_{jk}S_{i+M,l+M}(m+n)\psi(\epsilon n)\\ &+ \delta_{il}\ e_j(m+n)\psi(\epsilon n)\\ &+ \delta_{jk}\ \delta_{li}\ n\delta_{m+n,0}\psi(\epsilon n)\end{aligned}$$

Now taking $\epsilon \to 0$ we see that

$$\begin{aligned} A \quad & \to \delta_{jk}\ S_{i+M,l+M}(m+n) + \delta_{il}\ e_j(m+n) \\ & + \delta_{jk}\ \delta_{li}\ n\delta_{m+n,0} \end{aligned}$$

Thus we have verified 3.3(1).

We will now verify 3.3(2).

Consider

$$\begin{aligned} B &= [S_{k,l+M}(n), S_{i+M,j}(m)]\psi(\epsilon n) \\ &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\left[X_{s-\frac{1}{2}}(e_k)\varphi^{l^*}_{n-s+\frac{1}{2}}, X_{r-\frac{1}{2}}(-e_j)\varphi^{i}_{m-r+\frac{1}{2}}\right]\psi(\epsilon n) \end{aligned}$$

We will now use Lemma (2.8).

$$\begin{aligned} B &= \sum_{r,s\in\mathbb{Z}}\left[X_{s-\frac{1}{2}}(e_k), X_{r-\frac{1}{2}}(-e_j)\right]\varphi^{l^*}_{n-s+\frac{1}{2}}\varphi^{i}_{m-r+\frac{1}{2}}\psi(\epsilon n) \\ &- \sum_{r,s\in\mathbb{Z}} X_{r-\frac{1}{2}}(-e_j)X_{s-\frac{1}{2}}(e_k)\left[\varphi^{l^*}_{n-s+\frac{1}{2}}, \varphi^{i}_{m-r+\frac{1}{2}}\right]\psi(\epsilon n) \end{aligned}$$

We will now use 3.1(3) and Corollary 1.9(2)

$$\begin{aligned} B &= \delta_{kj}\sum_{r,s\in\mathbb{Z}}\delta_{r+s-1,0}\varphi^{l^*}_{n-s+\frac{1}{2}}\varphi^{i}_{m-r+\frac{1}{2}}\psi(\epsilon n) \\ &- \delta_{li}\sum_{r,s\in\mathbb{Z}} X_{r-\frac{1}{2}}(-e_j)X_{s-\frac{1}{2}}(e_k)\delta_{m+n-(r+s)+1,0}\psi(\epsilon n) \\ &= \delta_{kj}\sum_{r\in\mathbb{Z}}\varphi^{l^*}_{n+r-\frac{1}{2}}\varphi^{i}_{m-r+\frac{1}{2}}\ \psi(\epsilon n) \\ &- \delta_{li}\sum_{r\in\mathbb{Z}} X_{r-\frac{1}{2}}(-e_j)X_{m+n-r+\frac{1}{2}}(e_k)\psi(\epsilon n) \end{aligned}$$

We will now normal order the sums. For that we split first sum according $n+r-\frac{1}{2} \le m-r+\frac{1}{2}$ or $n+r-\frac{1}{2} > m-r+\frac{1}{2}$ and split the second sum according $r-\frac{1}{2} \le m+n-r+\frac{1}{2}$ *or* $r-\frac{1}{2} > m+n-r+\frac{1}{2}$

$$\begin{aligned} B &= \delta_{kj}\sum_{r\in\mathbb{Z}} :\varphi^{l^*}_{n+r-\frac{1}{2}}\ \varphi^{i}_{m-r+\frac{1}{2}}:\psi(\epsilon n) \\ &+ \delta_{kj}\delta_{il}\sum_{2r>m-n+1}\psi(\epsilon n)\delta_{m+n,0} \\ &- \delta_{il}\sum_{r\in\mathbb{Z}} :X_{r-\frac{1}{2}}(-e_j)X_{m+n-r+\frac{1}{2}}(e_k):\psi(\epsilon n) \\ &- \delta_{il}\delta_{kj}\sum_{2r>m+n+1}\psi(\epsilon n)\delta_{m+n,0} \end{aligned}$$

The second + the last term equals

$$\begin{aligned} &\delta_{kj}\delta_{il}\left(\sum_{r>m+\frac{1}{2}}\psi(\epsilon n)\delta_{m+n,0} - \sum_{r>\frac{1}{2}}\psi(\epsilon n)\delta_{m+n,0}\right) \\ &= -\delta_{kj}\delta_{il}m\delta_{m+n,0}\ \psi(\epsilon n) \end{aligned}$$

Note that

$$\begin{aligned} -\sum_{r\in\mathbb{Z}} :X_{r-\frac{1}{2}}(-e_j)X_{m+n-r+\frac{1}{2}}(e_k): \ &= \sum :X_{m+n-r+\frac{1}{2}}(e_k)X_{r-\frac{1}{2}}(-e_j): \\ &= e_k(m+n) \text{ if } j=k \\ &= F(e_k,-e_j)X_{m+n}(e_k-e_j)\ if\ j\neq k \end{aligned}$$

Thus as $\epsilon \to 0$

$$
\begin{aligned}
B \to \quad & \delta_{kj} S_{i+M,l+M}\ (m+n) + \delta_{li} F(e_k, -e_j) X_{m+n}(e_k - e_j)\ if\ k \neq j \\
B \to \quad & \delta_{kj} S_{i+M,l+M}(m+n) + \delta_{li} e_k(m+n) \\
& -\delta_{kj}\delta_{li} m \delta_{m+n,0}\ \ if\ k = j
\end{aligned}
$$

This proves 3.3(2).

We will now verify the first part of (R3). Consider for $i \neq j$

$$
\begin{aligned}
&[X_m(e_i - e_j), S_{k+M,l}(n)] = \textstyle\sum_{r\in\mathbb{Z}} [X_m(e_i - e_j), X_{r-\frac{1}{2}}(-e_l)\varphi^k_{n-r+\frac{1}{2}}] \\
&= \textstyle\sum_{r\in\mathbb{Z}} [X_m(e_i - e_j),\ X_{r-\frac{1}{2}}(-e_l)]\varphi^k_{m-r+\frac{1}{2}} \quad \text{(Using 2.1(3))} \\
&= -\delta_{il} \textstyle\sum_{r\in\mathbb{Z}} F(-e_l, e_i - e_j) X_{m+r-\frac{1}{2}}(-e_j)\varphi^k_{m-r+\frac{1}{2}} \quad \text{(Using Corollary 1.9(1))} \\
&= \delta_{il} F(e_j, e_i) S_{k+M,j}(m+n) \qquad \square
\end{aligned}
$$

We will now verify the second part of (R3). For $i \neq j$.

$$
\begin{aligned}
&[S_{k+M,l}(n), X_m(e_i - e_j) = \textstyle\sum_{r\in\mathbb{Z}} [X_{r-\frac{1}{2}}(-e_l)\varphi^k_{n-r+\frac{1}{2}},\ X_m(e_i - e_j)] \\
&= \textstyle\sum_{r\in\mathbb{Z}} [X_{r-\frac{1}{2}}(-e_l), X_m(e_i - e_j)]\varphi^k_{n-r+\frac{1}{2}} \quad \text{(Using 2.1(3))} \\
&= \delta_{il} \textstyle\sum_{r\in\mathbb{Z}} F(e_i, e_j) X_{m+r-\frac{1}{2}}(-e_j)\varphi^k_{n-r+\frac{1}{2}} \quad \text{(Using Corollary 1.9(1))} \\
&= \delta_{il} F(e_i, e_j) S_{k+M,j}(m+n)\ (Here\ and\ above\ we\ are\ using \\
&the\ property\ of\ the\ co-cycle\ (1.2).)
\end{aligned}
$$

We will now verify the third part of (R3).

$$
\begin{aligned}
&[e_i(m), S_{k+M,l}(n)] = \sum_{\gamma\in\mathbb{Z}} \left[e_i(m), X_{r-\frac{1}{2}}(-e_l)\varphi^k_{n-r+\frac{1}{2}} \right] \\
&= \sum_{r\in\mathbb{Z}} [e_i(m), X_{r-\frac{1}{2}}(-e_l)]\varphi^k_{n-r+\frac{1}{2}} \quad \text{(Using 2.7)} \\
&-\delta_{il} \sum_{r\in\mathbb{Z}} X_{r+m-\frac{1}{2}}(-e_l)\varphi^k_{n-r+\frac{1}{2}} \quad \text{(Using Corollary 1.9(3))} \\
&= -\delta_{il}\ S_{k+M,l}(m+n)
\end{aligned}
$$

We will now prove the last part of (R3)

$$
\begin{aligned}
&[S_{k+M,l}(n), e_i(m)] = \sum_{r\in\mathbb{Z}} [X_{r-\frac{1}{2}}(-e_l)\varphi^k_{n-r+\frac{1}{2}}, e_i(m)] \\
&= \sum_{r\in\mathbb{Z}} [X_{r-\frac{1}{2}}(-e_l), e_i(m)]\varphi^k_{n-r+\frac{1}{2}} \quad \text{(Using 2.7)} \\
&= \delta_{il} \sum_{r\in\mathbb{Z}} X_{m+r-\frac{1}{2}}(-e_l)\varphi^k_{n-r+\frac{1}{2}} \quad \text{(Using Corollary 1.9(3))} \\
&= \delta_{il}\ S_{k+M,l}(m+n)
\end{aligned}
$$

The relations $R4, R5$ and $R6$ are similar and much easier as the central term does not appear. We omit the details. The relations $R8, R9$ and $R10$ are

trivial to verify.

We will now verify $R2$ which is actually very standard. We could not find any decent proof which is suitable for our situation. Any How we thought for the sake of completeness, we offer a short proof.

Consider

$$\begin{aligned} D &= [S_{i+M,j+M}(m), S_{k+M,l+M}(n)]\psi(\epsilon n) \\ &= \sum_{\gamma,s\in\mathbb{Z}} \left[\varphi^i_{r-\frac{1}{2}}\varphi^{j^*}_{m-r+\frac{1}{2}}, \varphi^k_{s-\frac{1}{2}}\varphi^{l^*}_{n-s+\frac{1}{2}}\right]\psi(\epsilon n) \end{aligned}$$

We will be using (2.7) and (3.1). Recall that all φ operators are even. Normal ordering is not necessary as they differ by a central operator.

$$\begin{aligned} D &= \sum_{r,s\in\mathbb{Z}} \varphi^i_{r-\frac{1}{2}} \left[\varphi^{j^*}_{m-r+\frac{1}{2}}, \varphi^k_{s-\frac{1}{2}}\varphi^{l^*}_{n-s+\frac{1}{2}}\right]\psi(\epsilon n) \\ &+ \sum_{r,s\in\mathbb{Z}} \left[\varphi^i_{r-\frac{1}{2}}, \varphi^k_{s-\frac{1}{2}}\varphi^{l^*}_{n-s+\frac{1}{2}}\right]\varphi^{j^*}_{m-r+\frac{1}{2}}\psi(\epsilon n) \\ &= \sum_{r,s\in\mathbb{Z}} \varphi^i_{r-\frac{1}{2}} \left[\varphi^{j^*}_{m-r+\frac{1}{2}}, \varphi^k_{s-\frac{1}{2}}\right]\varphi^{l^*}_{n-s+\frac{1}{2}}\psi(\epsilon n) \\ &+ \sum_{r,s\in\mathbb{Z}} \varphi^k_{s-\frac{1}{2}} \left[\varphi^i_{r-\frac{1}{2}}, \varphi^{l^*}_{n-s+\frac{1}{2}}\right]\varphi^{j^*}_{m-r+\frac{1}{2}}\psi(\epsilon n) \end{aligned}$$

$$\begin{aligned} D &= \delta_{jk}\sum_{r,s\in\mathbb{Z}} \varphi^i_{r-\frac{1}{2}}\varphi^{l^*}_{n-s+\frac{1}{2}}\delta_{m-r+s,0}\ \psi(\epsilon n) \\ &- \delta_{il}\sum_{r,s\in\mathbb{Z}} \varphi^k_{s-\frac{1}{2}}\varphi^{j^*}_{m-r+\frac{1}{2}}\delta_{n-s+r,0}\ \psi(\epsilon n)\ \text{(Using (3.1))} \\ &= \delta_{jk}\sum_{r\in\mathbb{Z}} \varphi^i_{r-\frac{1}{2}}\varphi^{l^*}_{m+n-r+\frac{1}{2}}\ \psi(\epsilon n) \\ &- \delta_{il}\sum_{r\in\mathbb{Z}} \varphi^k_{n+r-\frac{1}{2}}\varphi^{j^*}_{m-r+\frac{1}{2}}\ \psi(\epsilon n) \end{aligned}$$

We need to normal order the sums. For this we need split the first sum accordingly to $r-\frac{1}{2} \le n+m-r+\frac{1}{2}$ or $r-\frac{1}{2} > n+m-r+\frac{1}{2}$ We split the second sum according to $n+r-\frac{1}{2} \le m-r+\frac{1}{2}$ or $n+r-\frac{1}{2} > m-r+\frac{1}{2}$

$$
\begin{aligned}
D &= \delta_{jk} \sum_{r \in \mathbb{Z}} : \ \varphi^i_{r-\frac{1}{2}} \ \varphi^{l^*}_{n+m-r+\frac{1}{2}} : \ \psi(\epsilon n) \\
&- \ \delta_{jk}\delta_{ii} \sum_{2r> \ n+m+1} \psi(\epsilon n)\delta_{m+n,0} \\
&- \ \delta_{il} \sum_{r \in \mathbb{Z}} : \ \varphi^k_{n+r-\frac{1}{2}} \ \varphi^{j^*}_{m-r+\frac{1}{2}} : \ \psi(\epsilon n) \\
&+ \ \delta_{jk} \ \delta_{li} \sum_{2r> \ m-n+1} \psi(\epsilon n) \ \delta_{m+n,0}
\end{aligned}
$$

The second term and fourth term equal to $-\delta_{jk} \ \delta_{il} \ m\psi(\epsilon n)$

So as $\epsilon \to 0$

$$
D \to \delta_{jk} \ S_{i+M,l+M}(m+n) - \delta_{il} \ S_{k+M,j+M}(m+n) - \delta_{jk} \ \delta_{il} \ m \ \delta_{m+n,0.}
$$

This completes the Proof of the Proportion 3.3

4 Toroidal Superalgebras

In this section we define toroidal superalgebras corresponding to $gl(M|N)$ and give representation using a bigger Fock space.

Let $\mathfrak{g}$ be a Lie superalgebras and let $(\ , \ )$ be a bilinear form on $\mathfrak{g}$ which supersymmetric, invariant and even. Fix a positive integer q and consider $A = \mathbb{C}[t_1^{\pm 1}, \cdots t_q^{\pm 1}]$ a Laurent polynomial ring in q commuting variables. Let $\overline{m} = (m_1, \cdots m_q) \in \mathbb{Z}^q$ and let $t^{\overline{m}} = t_1^{m_1} \cdots t_q^{m_q}$. Then $\mathfrak{g} \otimes A$ has a natural structure of Lie superalgebra. Let $X(\overline{m}) = X \otimes t^{\overline{m}} \in \mathfrak{g} \otimes A$.

Let $Z = \Omega_A/d_A$ be the space of differentials so that Ω_A is spanned by symbols $t^{\overline{m}}K_i, \overline{m} \in \mathbb{Z}^q, 1 \leq i \leq q$ and d_A is spanned by $\sum m_i t^{\overline{m}} K_i$. We now define Lie superalgebra structure on

$$
\mathfrak{g} \otimes A \oplus \Omega_A/d_A
$$

(4.1) $$[X(\overline{m}), Y(\overline{n})] = [X,Y](\overline{m}+\overline{n}) + (X,Y)d(t^{\overline{m}})t^{\overline{n}}$$

Where $$d(t^{\overline{m}})t^{\overline{n}} = \sum m_i \ t^{\overline{m}+\overline{n}} K_i$$

and $X, Y \in \mathfrak{g}, \overline{m}, \overline{n} \in \mathbb{Z}^q$.

Ω_A/d_A is central.

We call this algebra a toroidal superalgebra. See $[EZ]$ for more details.

Define $sl(M|N) = \{X \in gl(M|N) | strX = 0\}$ which is a Lie supersubalgebra of $gl(M|N)$

(4.2) Theorem (Theorem (4.7) of [IK]).

$sl(M|N) \otimes A \oplus \Omega_A/d_A$ is the universal central extension of $sl(M|N) \otimes A$ (If $M = N$, then we take $M > 2$)

The purpose of this section is to give representation of $gl(M|N) \otimes A \oplus \Omega_A/d_A$. In particular we give a representation for the universal central extension of $sl(M|N) \otimes A$.

We work with toroidal superalgebra $\tau = T \otimes A \otimes \Omega_A/d_A$ with bilinear form $(\ , \ )_T$ on T. The algebra T and the form $(\ , \ )_T$ are defined in section 2.

We will now write down the superbracket on τ using the superbracket on T given section 2 ($T1$ to $T10$)

$(ST1)$ $1 \leq i, j, k, l \leq M$

$i \neq j, k \neq l$

$$[T_{ij}(\overline{m}), T_{kl}(\overline{n})] = \begin{cases} 0 \text{ if } (\alpha_{ij}, \alpha_{kl}) \geq 0 \\ F(\alpha_{ij}, \alpha_{kl}) T_{il}(\overline{m} + \overline{n}) \text{ if } j = k, l \neq i, (\alpha_{ij}, \alpha_{kl}) = -1 \\ F(\alpha_{ij}, \alpha_{kl}) T_{kj}(\overline{m} + \overline{n}) \text{ if } l = i, j \neq k, (\alpha_{ij}, \alpha_{kl}) = -1 \\ F(\alpha_{ij}, \alpha_{kl})((T_{ii} - T_{jj})(\overline{m} + \overline{n}) + d(t^{\overline{m}})t^{\overline{n}}) \text{ if } l = i, j = k, (\alpha_{ij}, \alpha_{kl}) = -2 \end{cases}$$

$i = j, k \neq l$

$[T_{ii}(\overline{m}), T_{kl}(\overline{n})] = (e_i, \alpha_{kl}) T_{kl}(\overline{m} + \overline{n})$

$i = j, k = l$

$[T_{ii}(\overline{m}), T_{kk}(\overline{n})] = \delta_{ik} \ d(t^{\overline{m}})t^{\overline{n}})$

$(ST2)$ $\quad 1 \leq i,j,k,l \leq N$

$$[T_{i+M,j+M}(\overline{m}), T_{k+M,l+M}(\overline{n})] = \delta_{jk}\ T_{i+M,l+M}\ (\overline{m}+\overline{n})$$
$$-\delta_{li}\ T_{k+M,j+M}\ (\overline{m}+\overline{n}) - \delta_{jk}\ \delta_{li}\ d(t^{\overline{m}})t^{\overline{n}}$$

$(ST3)$ $\quad 1 \leq i,j,l \leq M, 1 \leq k \leq N, i \neq j$

$$[T_{ij}(\overline{m}), T_{k+M,l}(\overline{n})] = \delta_{il}\ F(e_j,e_i)T_{k+M,j}(\overline{m}+\overline{n})$$
$$[T_{k+M,l}(\overline{n}), T_{ij}(\overline{m})] = \delta_{il}\ F(e_j,e_i)T_{k+M,j}(\overline{m}+\overline{n})$$
$$[T_{ii}(\overline{m}), T_{k+M,l}(\overline{n})] = -\delta_{il}\ T_{k+M,l}(\overline{m}+\overline{n})$$
$$[T_{k+M,l}(\overline{n}), T_{ii}(\overline{m})] = \delta_{il}\ T_{k+M,l}(\overline{m}+\overline{n})$$

$(ST4)$ $\quad 1 \leq i,j,k \leq M, 1 \leq l \leq N$

$$[T_{ij}(\overline{m}), T_{k,l+M}(\overline{n})] = \delta_{jk}\ F(e_i,e_j)T_{i,l+M}(\overline{m}+\overline{n})$$
$$[T_{k,l+M}(\overline{n}), T_{ij}(\overline{m})] = -\delta_{jk}\ F(e_i,e_j)T_{i,l+M}(\overline{m}+\overline{n})$$

$(ST5)$ $\quad 1 \leq i,j,k \leq N, 1 \leq l \leq M$

$$[T_{i+M,j+M}(\overline{m}), T_{k+M,l}(\overline{n})] = \delta_{jk}\ T_{i+M,l}\ (\overline{m}+\overline{n})$$
$$[T_{k+M,l}(\overline{n}), T_{i+M,j+M}(\overline{m})] = -\delta_{jk}\ T_{i+M,l}(\overline{m}+\overline{n})$$

$(ST6)$ $\quad 1 \leq i,j,l \leq N, 1 \leq k \leq M$

$$[T_{i+M,j+M}(\overline{m}), T_{k,l+M}(\overline{n})] = -\delta_{li}\ T_{k,j+M}\ (\overline{m}+\overline{n})$$
$$[T_{k,l+M}(\overline{n}), T_{i+M,j+M}(\overline{m})] = \delta_{li}\ T_{k,j+M}(\overline{m}+\overline{n})$$

$(ST7)$ $\quad 1 \leq i,l \leq N, 1 \leq j,k \leq M$

$$[T_{i+M,j}(\overline{m}), T_{k,l+M}(\overline{n})] = \delta_{jk}\ T_{i+M,l+M}(\overline{m}+\overline{n}) + F(e_k,e_j)\delta_{li}\ T_{kj}(\overline{m}+\overline{n})$$
$$-\delta_{jk}\ \delta_{li}\ d(t^{\overline{m}})t^{\overline{n}}$$
$$[T_{k,l+M}(\overline{n}), T_{i+M,j}(\overline{m})] = \delta_{jk}\ T_{i+M,l+M}(\overline{m}+\overline{n}) + F(e_k,e_j)\delta_{li}\ T_{kj}(\overline{m}+\overline{n})$$
$$+\delta_{jk}\ \delta_{li}\ d(t^{\overline{n}})t^{\overline{m}}$$

$(ST8)$ $\quad 1 \leq i,j \leq M, 1 \leq k,l \leq N$

$$[T_{ij}(\overline{m}), T_{k+M,l+M}(\overline{n})] = [T_{k+M,l+M}(\overline{n}), T_{ij}(\overline{m})] = 0$$

$(ST9)$ $\quad 1 \leq i,k \leq N, 1 \leq j,l \leq M$

$$[T_{i+M,j}(\overline{m}), T_{k+M,l}(\overline{n})] = [T_{k+M,l}(\overline{n}), T_{i+M,j}(\overline{m})] = 0$$

$(ST10)$ $\quad 1 \leq i,k \leq M, 1 \leq j,l \leq N$

$$[T_{i,j+M}(\overline{m}), T_{k,l+M}(\overline{n})] = [T_{k,l+M}(\overline{n}), T_{i,j+M}(\overline{m})] = 0$$

(4.3) We will now extend the vertex operator construction to bigger lattice to accommodate the toroidal case. Let $\overline{\Gamma}$ be a free $\mathbb{Z}$ module on generators $e_1, e_2, \cdots e_M, \delta_1, \cdots, \delta_{q-1}, d_1, \cdots d_{q-1}$. (These δ_i's are not to be confused with δ function defined earlier. These δ_i's always come with an index).Define a non-degenerate symmetric bilinear form on $\overline{\Gamma}$ by

$(e_i, e_j) = \delta_{ij}, (e_i, \delta_j) = (e_i, d_j) = (\delta_i, \delta_j) = (d_i, d_j) = 0, (\delta_i, d_j) = \delta_{ij}$

Let $\overline{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} \overline{\Gamma}$ and Let $\overset{\triangle}{\overline{\mathfrak{h}}} = \oplus_{k \in \mathbb{Z}} \overline{\mathfrak{h}}(k) \oplus \mathbb{C}K$

where $\overline{\mathfrak{h}}(k)$ is a copy of $\overline{\mathfrak{h}}$. The Lie algebra structure on $\overset{\triangle}{\overline{\mathfrak{h}}}$ is given by

$$[\alpha(k), \beta(m)] = k(\alpha, \beta)\delta_{k+m,0}K$$

Let Q be the sublattice of $\overline{\Gamma}$ spanned by $e_1, \cdots e_M, \delta_1, \cdots, \delta_{q-1}$.

Recall that the lattice Γ spanned by $e_1, \cdots e_M$ is defined earlier and hence contained in $\overline{\Gamma}$.

Recall that the Co-cycle F is define on Γ. We now extend the Co-cycle F to Q by

$$F(e_i, \delta_k) = F(\delta_k, e_i) = F(\delta_k, \delta_l) = 1$$

We further extend F to be a bimultiplicative map $F : Q \times \overline{\Gamma} \to \{\pm 1\}$ is any convenient way.

Consider the group algebra $\mathbb{C}[\overline{\Gamma}]$ and $\mathbb{C}[Q]$. Make $\mathbb{C}[\overline{\Gamma}]$ and $\mathbb{C}[Q]$ module be defining.

$$e^{\alpha}.e^{\gamma} = F(\alpha, \gamma)e^{\alpha+\gamma}, \alpha \in Q, \gamma \in \overline{\Gamma}$$

Let $\overline{\mathfrak{h}}_- = \oplus_{k<0} \overline{\mathfrak{h}}(k)$ and consider the Fock space $V[\overline{\Gamma}] = \mathbb{C}[\overline{\Gamma}] \otimes S(\overline{\mathfrak{h}}_-)$ where $S(\overline{\mathfrak{h}}_-)$ is the symmetric algebra of $\overline{\mathfrak{h}}_-$.

(4.4) Notation.

Let $\underline{m} = (m_1, \cdots m_{q-1}) \in \mathbb{Z}^{q-1}$

So that $\overline{m} = (\underline{m}, m_q) \in \mathbb{Z}^q$.

Let $\delta_{\underline{m}} = \sum_{i=1}^{q-1} m_i \; \delta_i$

Define vertex operator as earlier for $\alpha \in Q$

$$Y(\alpha, z) = e^{\alpha} z^{\alpha(0)} \; exp \; T_-(\alpha, z) \; exp \; T_+(\alpha, z)$$

When $T_+(\alpha, z)$ are similarly defined as in section 1. There is a natural $\mathbb{Z}_2$-gradation on $\overline{\Gamma}$ comeing from the norm of α in $\overline{\Gamma}$.

Write $\overline{\Gamma} = \overline{\Gamma}_{\bar{0}} \oplus \overline{\Gamma}_{\bar{1}}$

For $\alpha \in \overline{\Gamma}_{\bar{0}}$ let $X(\alpha, z) = z^{\frac{(\alpha,\alpha)}{2}} Y(\alpha, z)$

For $\alpha \in \overline{\Gamma}_{\bar{1}}$ let $X(\alpha, z) = Y(\alpha, z)$

In particular

$$Y(\delta_{\underline{m}}, z) = X(\delta_{\underline{m}}, z) = \sum_{n \in \mathbb{Z}} X_n(\delta_{\underline{m}}) z^{-n}$$

Write $$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1}, \alpha \in \overline{\Gamma}.$$

The following is very obvious ($\alpha \in Q$)

$$X(\alpha, z) X(\delta_{\underline{m}}, z) = X(\alpha + \delta_{\underline{m}}, z)$$

as $$(\alpha, \delta_{\underline{m}}) = 0 \text{ and } F(\alpha, \delta_{\underline{m}}) = 1.$$

It is clear that Lemma 1.8 holds for $\alpha, \beta, \in Q$.

We now consider much bigger Fock space.

$$V[\overline{\Gamma}] \otimes \mathfrak{F}$$

The operators $X(\alpha, z),\ \alpha \in Q$ acts on the first component and the operators φ_i's and φ_i^* act on the second component. In particular they commute.

(4.5) Recall the S operators and $\alpha_{ij} = e_i - e_j$ from earlier section.

Let

$$\begin{aligned} S_{ij}^{\underline{m}}(z) = & \quad S_{ij}(z) X(\delta_{\underline{m}}, z) \\ = & \sum_{n \in \mathbb{Z}} S_{ij}^{\underline{m}}(n) z^{-n-1} \end{aligned}$$

so that $S_{ij}^{\underline{m}}(n) = \sum_{k \in \mathbb{Z}} S_{ij}(k) X_{n-k}(\delta_{\underline{m}})$

Let

$$\alpha(z) X(\delta_{\underline{m}}, z) = \sum_{n \in \mathbb{Z}} T_n^{\alpha}(\delta_{\underline{m}}) z^{-n-1}$$

so that $T_n^{\alpha}(\delta_{\underline{m}}) = \sum_{k \in \mathbb{Z}} \alpha(k) X_{n-k}(\delta_{\underline{m}})$

We now state and Prove the main theorem of this paper

Theorem (4.6) The operators X and S define a representation of $\tau = T \otimes A \otimes \Omega_A/d_n$ on the space $V(\overline{\Gamma}) \otimes \mathfrak{F}$ by the following correspondence

$1 \leq i, j \leq M$

$i \neq j \qquad T_{ij} \otimes t^{\overline{m}} \mapsto X_{m_q}(\alpha_{ij} + \delta_{\underline{m}})$

$i = j \qquad T_{ii} \otimes t^{\overline{m}} \mapsto T^{e_i}_{m_q}(\delta_{\underline{m}})$

If one of i or $j > M$

$T_{ij} \otimes t^{\overline{m}} \mapsto S^{\underline{m}}_{ij}(m_q)$

$d(t^{\overline{m}})t^{\overline{n}} \mapsto X_{m_q+n_q}(\delta_{\underline{m}+\underline{n}}) + m_q \; T^{\delta_{\underline{m}}}_{m_q+n_q}(\delta_{\underline{m}+\underline{n}})$

We need the following

Lemma 4.7

$$\begin{aligned} (1) \quad D_z z X(\delta_{\underline{m}}, z) &= z\delta_{\underline{m}}(z)X(\delta_{\underline{m}}, z) + X(\delta_{\underline{m}}, z) \\ (2) \quad D_z X(\delta_{\underline{m}}, z) &= \delta_{\underline{m}}(z)X(\delta_{\underline{m}}, z) \end{aligned}$$

Proof Follows from Lemma(4.6) of $[EM]$ just note that $\alpha(z)$ in our paper and $\alpha(z)$ in $[EM]$ differ by a factor of z.

Lemma 4.8

$$\begin{aligned} (1) \quad & D_z\delta(z/w).zX(\delta_{\underline{m}}, z)X(\delta_{\underline{n}}, w) \\ & = wX(\delta_{\underline{m}+\underline{n}}, w)D_z \; \delta(z/w) \\ & -w \; \delta_{\underline{m}}(w)X(\delta_{\underline{m}+\underline{n}}, w)\delta(z/w) \\ & -X(\delta_{\underline{m}+\underline{n}}, w)\delta(z/w) \end{aligned}$$

$$\begin{aligned} (2) \quad & D_z\delta(z/w)X(\delta_{\underline{m}}, z)X(\delta_{\underline{n}}, w) \\ & = X(\delta_{\underline{m}+\underline{n}}, w)D_z\delta(z/w) \\ & -\delta_{\underline{m}}(w)X(\delta_{\underline{m}+\underline{n}}, w)\delta(z/w) \end{aligned}$$

Proof Follows from Lemma 1.5(3) and Lemma (4.7)

Lemma 4.9

$$T^{\delta_{\underline{m}}}_{m_q}(\delta_{\underline{m}}) + m_q \; X_{m_q}(\delta_{\underline{m}}) = 0$$

Follows from Lemma 3.13 of $[EM]$.

(4.10) We have noted that $d(t^{\overline{m}})t^{\overline{n}} = -d(t^{\overline{n}})t^{\overline{m}}$. Because of the Lemma 4.9

this can be verified in the statement of the Theorem (4.6).

Proof of the Theorem (4.6).

It is sufficient to verify the following relation among the X and S operators.

$(SR1)$ $\quad 1 \leq i, j, k, l \leq M$

(1) $\quad i \neq j, k \neq l$

$$[X(\alpha_{ij}, z)X(\delta_{\underline{m}}, z), X(\alpha_{kl}, w)X(\delta_{\underline{n}}, w] = o \text{ if } (\alpha_{ij}, \alpha_{kl}) \geq 0$$
$$= F(\alpha_{ij}, \alpha_{kl})X(\alpha_{ij} + \alpha_{kl}, z)X(\delta_{\underline{m}+\underline{n}}, z)\delta(z/w) \text{ if } (\alpha_{ij}, \alpha_{kl}) = -1$$
$$= F(\alpha_{ij}, -\alpha_{ij})\alpha_{ij}(w)wX(\delta_{\underline{m}+\underline{n}}, w)\delta(z/w)$$
$$-F(\alpha_{ij}, -\alpha_{ij})(wX(\delta_{\underline{m}+\underline{n}}, w)\ D_z\delta(z/w)$$
$$-w\ \delta_{\underline{m}}(w)X(\delta_{\underline{m}+\underline{n}}, w)\delta(z/w)$$
$$-X(\delta_{\underline{m}+\underline{n}}, w)\delta(z/w)) \text{ if } \ (\alpha_{ij}, \alpha_{kl}) = -2$$

(2) $\quad i = j, k \neq l$

$$[e_i(z)X(\delta_{\underline{m}}, z), X(\alpha_{kl}, w)X(\delta_{\underline{n}}, w]$$
$$= (e_i, \alpha_{kl})w^{-1}X(\alpha_{kl}, w)X(\delta_{\underline{m}+\underline{n}}, w)\delta(z/w)$$

(3) $\quad i = j, k = l$

$$= [e_i(z)X(\delta_{\underline{m}}, z), e_k(w)X(\delta_{\underline{n}}, w)]$$
$$= -\delta_{kl}\ w^{-1}X(\delta_{\underline{m}+\underline{n}}, w)D_z\ \delta(z/w)$$
$$+\delta_{kl}\ w^{-1}\ \delta_{\underline{m}}(w)X(\delta_{\underline{m}+\underline{n}}, w)\delta(z/w)$$

$(SR2)$ $\quad 1 \leq i, j, k, l \leq N$

$$[S^{\underline{m}}_{i+M,j+M}(z), S^{\underline{n}}_{k+M,l+M}(w)]$$
$$= \delta_{jk}\ S^{\underline{m}+\underline{n}}_{i+M,l+M}(w)w^{-1}\ \delta(z/w)$$
$$-\delta_{li}\ S^{\underline{m}+\underline{n}}_{k+M,l+M}(w)w^{-1}\ \delta(z/w)$$
$$+\delta_{jk}\delta_{li}(w^{-1}X(\delta_{\underline{m}+\underline{n}}, w)\ D_z\delta(z/w)$$
$$-w^{-1}\delta_{\underline{m}}(w)X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w))$$

$(SR3)\quad 1 \le i, l \le M, 1 \le k \le N$

(1) $\quad i \ne j$

$$[X(\alpha_{ij}, z)X(\delta_{\underline{m}}, z), S^{\underline{n}}_{k+M,l}(w)]$$
$$= \delta_{il}\ F(e_j, e_i) S^{\underline{m}+\underline{n}}_{k+M,j}(z)\ \delta(z/w)$$
$$[S^{\underline{n}}_{k+M,l}(z), X(\alpha_{ij}, w)X(\delta_{\underline{m}}, w)]$$
$$= \delta_{il}\ F(e_i, e_j) S^{\underline{m}+\underline{n}}_{k+M,l}(w)\ \delta(z/w)$$

(2) $\quad i = j$

$$[e_i(z)X(\delta_{\underline{m}}, z), S^{\underline{n}}_{k+M,l}(w)]$$
$$= -\delta_{il}\ S^{\underline{m}+\underline{n}}_{k+M,l}(w) w^{-1}\ \delta(z/w)$$
$$[S^{\underline{n}}_{k+M,l}(z), e_i(w)X(\delta_{\underline{m}}, w)]$$
$$= \delta_{il}\ S^{\underline{m}+\underline{n}}_{k+M,l}(w) w^{-1}\ \delta(z/w)$$

$(SR4)\quad 1 \le i, j, k \le M, 1 \le l \le N$

(1) $\quad i \ne j$

$$[X(\alpha_{ij}, z)X(\delta_{\underline{m}}, z), S^{\underline{n}}_{k,l+M}(w)]$$
$$= \delta_{jk}\ F(e_i, e_j) S^{\underline{m}+\underline{n}}_{i,l+M}(w)\ \delta(z/w)$$
$$[S^{\underline{n}}_{k,l+M}(z), X(\alpha_{ij}, w)X(\delta_{\underline{m}}, w)]$$
$$= -\delta_{jk}\ F(e_i, e_j) S^{\underline{m}+\underline{n}}_{i,l+M}(w)\ \delta(z/w)$$

(2) $\quad i = j$

$$[e_i(z)X(\delta_{\underline{m}}, z), S^{\underline{n}}_{k,l+M}(w)]$$
$$= \delta_{ik}\ S^{\underline{m}+\underline{n}}_{i,l+M}(w) w^{-1}\ \delta(z/w)$$
$$[S^{\underline{n}}_{k,l+M}(z), e_i(w)X(\delta_{\underline{m}}, w)]$$
$$= -\delta_{ik}\ S^{\underline{m}+\underline{n}}_{i,l+M}\ w^{-1}\ \delta(z/w)$$

$(SR5)\quad 1 \le i, j, k \le N, 1 \le l \le M$

$$[S^{\underline{m}}_{i+M,j+M}(z), S^{\underline{n}}_{k+M,l}(w)]$$
$$= \delta_{jk}\ S^{\underline{m}+\underline{n}}_{i+M,l}(w) w^{-1}\ \delta(z/w)$$
$$[S^{\underline{n}}_{k+M,l}(z), S^{\underline{m}}_{i+M,j+M}(w)]$$
$$= -\delta_{jk}\ S^{\underline{m}+\underline{n}}_{i+M,l}\ w^{-1}\ \delta(z/w)$$

$(SR6)$ $1 \le i, j, l \le N, 1 \le k \le M$

$$[S^{\underline{m}}_{i+M,j+M}(z), S^{\underline{n}}_{k,l+M}(w)] = -\delta_{li}\ S^{\underline{m}+\underline{n}}_{k,j+M}(w) w^{-1}\ \delta(z/w)$$

$$[S^{\underline{n}}_{k,l+M}(z), S^{\underline{m}}_{i+M,j+M}(w)] = \delta_{li}\ S^{\underline{m}+\underline{n}}_{k,j+M}(w) w^{-1}\ \delta(z/w)$$

$(SR7)$ $1 \le i, l \le N, 1 \le j, k \le M$

$k \neq j$

$$[S^{\underline{m}}_{i+M,j}(z), S^{\underline{n}}_{k,l+M}(w)]$$
$$= \delta_{jk}\ S^{\underline{m}+\underline{n}}_{i+M,l+M}(w) w^{-1}\ \delta(z/w)$$
$$+\delta_{li}\ F(e_k, e_j) X(\alpha_{kj} + \delta_{\underline{m}+\underline{n}}, w) w^{-2}\ \delta(z/w)$$
$$+\delta_{jk}\delta_{li}(w^{-1} X(\delta_{\underline{m}+\underline{n}}, w)\ D_z \delta(z/w)$$
$$-w^{-1}\delta_{\underline{m}}(w) X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w))$$

$k \neq j$

$$[S^{\underline{n}}_{k,l+M}(z), S^{\underline{m}}_{i+M,j}(w)]$$
$$= \delta_{jk}\ S^{\underline{m}+\underline{n}}_{i+M,l+M} w^{-1}\ \delta(z/w)$$
$$+\delta_{li}\ F(e_k, e_j) X(\alpha_{kj} + \delta_{\underline{m}+\underline{n}}, w) w^{-2}\ \delta(z/w)$$
$$-\delta_{jk}\delta_{li}(w^{-1} X(\delta_{\underline{m}+\underline{n}}, w)\ D_z \delta(z/w)$$
$$-w^{-1}\delta_{\underline{n}}(w) X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w))$$

$j = k$

$$[S^{\underline{m}}_{i+M,j}(z), S^{\underline{n}}_{k,l+M}(w)]$$
$$= \delta_{jk}\ S^{\underline{m}+\underline{n}}_{i+M,l+M}(w) w^{-1}\ \delta(z/w)$$
$$+\delta_{li}\ F(e_k, e_j) e_j(w) w^{-1} X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w)$$
$$+\delta_{jk}\ \delta_{li}\ w^{-1} X(\delta_{\underline{m}+\underline{n}}, w)\ D_z \delta(z/w)$$
$$-\delta_{jk}\ \delta_{li}\ w^{-1}\ \delta_{\underline{m}}(w) X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w)$$

$j = k$

$$[S^{\underline{n}}_{k,l+M}(z), S^{\underline{m}}_{i+M,j}(w)]$$
$$= \delta_{jk}\ S^{\underline{m}+\underline{n}}_{i+M,l+M}(w) w^{-1}\ \delta(z/w)$$
$$+\delta_{li}\ F(e_k, e_j) e_j(w) w^{-1} X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w)$$
$$-\delta_{jk}\ \delta_{li}\ w^{-1} X(\delta_{\underline{m}+\underline{n}}, w)\ D_z \delta(z/w)$$
$$+\delta_{jk}\ \delta_{li}\ w^{-1}\ \delta_{\underline{n}}(w) X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w)$$

$$(SR8) \quad 1 \le i,j \le M, 1 \le k,l \le N$$

$$[X(\alpha_{ij},z)X(\delta_{\underline{m}},z), S^{\underline{n}}_{k+M,l+M}(w)] =$$

$$[S^{\underline{n}}_{k+M,l+M}(z), X(\alpha_{ij},w)X(\delta_{\underline{m}},w)] = 0$$

Note that for $i=j$, $X(\alpha_{ij},z)$ has to be replaced by $e_i(z)$.

$$(SR9) \quad 1 \le i,k \le N, 1 \le j,l \le M$$

$$[S^{\underline{m}}_{i+M,j}(z), S^{\underline{n}}_{k+M,l}(w)] =$$

$$[S^{\underline{m}}_{k+M,l}(z), S^{\underline{n}}_{i+M,j}(w)] = 0$$

$$(SR10) \quad 1 \le i,k \le M, 1 \le j,l \le N$$

$$[S^{\underline{m}}_{i,j+M}(z), S^{\underline{n}}_{k,l+M}(w)]$$

$$= [S^{\underline{n}}_{k,l+M}(z), S^{\underline{m}}_{i,j+M}(w)] = 0$$

When we expand the above infinite series in components form we will see that the relations $SR1$ to $SR10$ will be equal to $ST1$ to $ST10$. In order to verify the relations $SR1$ to $SR10$, we need to rewrite the relations $R1$ to $R10$ in the infinite series form. Thus we have the following.

$(S1)$ $\quad 1 \le i,j,k,l \le M$

(1) $\quad i \ne j, k \ne l$

$$[X(\alpha_{ij},z), X(\alpha_{kl},w)] = 0 \text{ if } (\alpha_{ij},\alpha_{kl}) \ge 0$$

$$= F(\alpha_{ij},\alpha_{kl})X(\alpha_{ij}+\alpha_{kl},z)\delta(z/w) \text{ if } (\alpha_{ij},\alpha_{kl}) = -1$$

$$= F(\alpha_{ij},-\alpha_{ij})(\alpha_{ij}(z)z\ \delta(z/w) - z\ D_z\delta(z/w)) \text{ if } \ (\alpha_{ij},\alpha_{kl}) = -2$$

(2) $\quad i = j, k \ne l$

$$[e_i(z), X(\alpha_{kl},w)] = z^{-1}(e_i,\alpha_{kl})X(\alpha_{kl},z)\ \delta(z/w)$$

(3) $\quad i = j, k = l$

$$[e_i(z), e_k(w)] = -\delta_{ik}\ w^{-1}\ D_z\ \delta(z/w)$$

$(S2)\quad 1 \leq i, j, k, l \leq N$

$$[S_{i+M,j+M}(z), S_{k+M,l+M}(w)] =$$
$$\delta_{jk}\ S_{i+M,l+M}(z)\ w^{-1}\ \delta(z/w)$$
$$-\delta_{li}\ S_{k+M,j+M}(z)\ w^{-1}\ \delta(z/w)$$
$$+\delta_{jk}\ \delta_{li}\ w^{-1}\ \ D_z\delta(z/w)$$

$(S3)\quad 1 \leq i, j, l \leq M, 1 \leq k \leq N$

(1) $i \neq j$

$$[X(\alpha_{ij}, z), S_{k+M,l}(w)]$$
$$= \delta_{il}\ F(e_j, e_i)S_{k+M,j}(z)\ \delta(z/w)$$
$$[S_{k+M,l}(z), X(\alpha_{ij}, w)]$$
$$= \delta_{il}\ F(e_i, e_j)S_{k+M,j}(z)\ \delta(z/w)$$

(2) $i = j$

$$[e_i(z), S_{k+M,l}(w)]$$
$$= -\delta_{il}\ S_{k+M,l}(z)\ w^{-1}\ \delta(z/w)$$
$$[S_{k+M,l}(z), e_i(z)]$$
$$= \delta_{il}\ S_{k+M,l}(z)\ w^{-1}\ \delta(z/w)$$

$(S4)\quad 1 \leq i, j, k \leq M, 1 \leq l \leq N$

(1) $i \neq j$

$$[X(\alpha_{ij}, z), S_{k,l+M}(w)]$$
$$= \delta_{jk}\ F(e_i, e_j)S_{i,l+M}(z)\ \delta(z/w)$$
$$[S_{k,l+M}(z), X(\alpha_{ij}, z)]$$
$$= -\delta_{jk}\ F(e_i, e_j)S_{i,l+M}\ (z)\ \delta(z/w)$$

(2) $i = j$

$$[e_i(z), S_{k,l+M}(w)]$$
$$= \delta_{ik}\ S_{i,l+M}(z)\ w^{-1}\ \delta(z/w)$$
$$[S_{k,l+M}(z), e_i(w)]$$
$$= -\delta_{ik}\ S_{i,l+M}(z)\ w^{-1}\ \delta(z/w)$$

$(S5)\quad 1 \leq i, j, k \leq N, 1 \leq l \leq M$

$$[S_{i+M,j+M}(z), S_{k+M,l}(w)]$$
$$= \delta_{jk}\ S_{i+M,l}(z)\ w^{-1}\ \delta(z/w)$$
$$[S_{k+M,l}(z), S_{i+M,j+M}(w)]$$
$$= -\delta_{jk}\ S_{i+M,l}(z)\ w^{-1}\ \delta(z/w)$$

$(S6)\quad 1 \leq i, j, l \leq N, 1 \leq k \leq M$

$$[S_{i+M,j+M}(z), S_{k,l+M}(w)]$$
$$= -\delta_{li}\ S_{k,j+M}(z)\ w^{-1}\ \delta(z/w)$$
$$[S_{k,l+M}(z), S_{i+M,j+M}(w)]$$
$$= \delta_{li}\ S_{k,j+M}(z)\ w^{-1}\ \delta(z/w)$$

$(S7)\quad 1 \leq i, l \leq N, 1 \leq j, k \leq M$

(1) $\quad j \neq k$

$$[S_{i+M,j}(z), S_{k,l+M}(w)]$$
$$= \delta_{jk}\ S_{i+M,l+M}(z)\ w^{-1}\ \delta(z/w)$$
$$+\delta_{li}\ F(e_k, e_j) X(\alpha_{kj}, z) z^{-1} w^{-1}\ \delta(z/w)$$
$$+\delta_{kj}\ \delta_{li}\ w^{-1}\ D_z \delta(z/w)$$

$j = k$

$$[S_{i+M,j}(z), S_{k,l+M}(w)]$$
$$= \delta_{jk}\ S_{i+M,l+M}(z) w^{-1}\ \delta(z/w)$$
$$+\delta_{li}\ F(e_k, e_j) e_j\ (z)\ w^{-1}\ \delta(z/w)$$
$$+\delta_{kj}\ \delta_{li}\ w^{-1}\ D_z \delta(z/w)$$

(2) $\quad j \neq k$

$$[S_{k,l+M}(z), S_{i+M,j}(w)]$$
$$= \delta_{jk}\ S_{i+M,l+M}(z) w^{-1}\ \delta(z/w)$$
$$+\delta_{li}\ F(e_k, e_j) X(\alpha_{kj}, z) z^{-1} w^{-1}\ \delta(z/w)$$
$$-\delta_{jk}\ \delta_{li}\ w^{-1}\ D_z \delta(z/w)$$

$j = k$

$[S_{k,l+M}(z), S_{i+M,j}(w)]$

$= \delta_{jk}\ S_{i+M,l+M}(z)\ w^{-1}\ \delta(z/w)$

$+\delta_{li}\ F(e_k, e_j)e_j(z)w^{-1}\ \delta(z/w)$

$-\delta_{jk}\ \delta_{li}\ w^{-1}\ D_z\delta(z/w)$

$(S8)$ $1 \leq i, j \leq M, 1 \leq k, l \leq N$

$[X(\alpha_{ij}, z), S_{k+M,l+M}(w)] = [S_{k+M,l+M}(z), X(\alpha_{ij}, w)] = 0$

Note that for $i = j$, $X(\alpha_{ij}, z)$ has to be replaced by $e_i(z)$.

$(S9)$ $1 \leq i, k \leq N, 1 \leq j, l \leq M$

$[S_{i+M,j}(z), S_{k+M,l}(w)] = [S_{k+M,l}(z), S_{i+M,j}(w)] = 0$

$(S10)$ $1 \leq i, k \leq M, 1 \leq j, l \leq N$

$[S_{i,j+M}(z), S_{k,l+M}(w)] = [S_{k,l+M}(z), S_{i,j+M}(w)] = 0$

It is now a simple matter to verify $SR1$ to $SR10$ by using Lemma 4.7 and 4.8 and the relations $S1$ to $S10$. We will only verify the third part of $SR1$ and $SR7$ which is more complex then others. The rest of the relations can be verified similarly.

Recall that the operators $X(\delta_{\underline{m}}, z)$ commute with all operators. To verify $SR1$ (third part) consider

$$\begin{aligned}
&[e_i(z)X(\delta_{\underline{m}}, z), e_k(w)X(\delta_{\underline{n}}, w)] \\
&= [e_i(z), e_k(w)]X(\delta_{\underline{m}}, z)X(\delta_{\underline{n}}, w) \\
&= -\delta_{ki}\ w^{-1}D_z\ \delta(z/w)X(\delta_{\underline{m}}, z)X(\delta_{\underline{n}}, w)\ (by\ S1\ ) \\
&= -\delta_{ki}\ w^{-1}X(\delta_{\underline{m}+\underline{n}}, w)D_z\ \delta(z/w) \\
&+\delta_{ki}\ w^{-1}\ \delta_{\underline{m}}(w)X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w)\ (by\ Lemma\ 4.8(2)\ )
\end{aligned}$$

Which is precisely $SR1$ (3rd part)

We will now verify the first and third part of $SR7$. Consider for $k \neq j$

$$
\begin{aligned}
&[S^{\underline{m}}_{i+M,j}(z), S^{\underline{n}}_{k,l+M}(w)] \\
&= [S_{i+M,j}(z)X(\delta_{\underline{m}}, z), S_{k,l+M}(w)X(\delta_{\underline{n}}, w)] \ (by\ 4.5\ ) \\
&= [S_{i+M,j}(z), S_{k,l+M}(w)]\ X(\delta_{\underline{m}}, z)\ X(\delta_{\underline{n}}, w) \\
&= \delta_{jk}\ S_{i+M,l+M}(z)w^{-1}\ \delta(z/w)\ X(\delta_{\underline{m}}, z)\ X(\delta_{\underline{n}}, w) \\
&+\delta_{li}\ F(e_k, e_j)X(\alpha_{kj}, z)z^{-1}w^{-1}\ \delta(z/w)X(\delta_{\underline{m}}, z)\ X(\delta_{\underline{n}}, w) \\
&+\delta_{jk}\ \delta_{li}\ w^{-1}\ D_z\delta(z/w)X(\delta_{\underline{m}}, z)\ X(\delta_{\underline{n}}, w)\ (by\ S7\ ) \\
&= \delta_{jk}\ S_{i+M,l+M}(w)\ w^{-1}X(\delta_{\underline{m}}, w)\ X(\delta_{\underline{n}}, w)\ \delta(z/w) \\
&+\delta_{li}\ F(e_k, e_j)X(\alpha_{kj}, w)w^{-2}\ X(\delta_{\underline{m}}, w)\ X(\delta_{\underline{n}}, w)\ \delta(z/w) \\
&+\delta_{jk}\delta_{li}(w^{-1}X(\delta_{\underline{m}+\underline{n}}, w)\ D_z\delta(z/w) \\
&-w^{-1}\delta_{\underline{m}}(w)X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w))\ \ (By\ Lemma\ 4.7\ and\ 4.8\ )
\end{aligned}
$$

for $k = j$

$$
\begin{aligned}
&[S^{\underline{m}}_{i+M,j}(z), S^{\underline{n}}_{k,l+M}(w)] = [S_{i+M,j}(z)X(\delta_{\underline{m}}, z), S_{k,l+M}(w)X(\delta_{\underline{n}}, w)] \ (by\ 4.5\ ) \\
&= [S_{i+M,j}(z), S_{k,l+M}(w)]\ X(\delta_{\underline{m}}, z)\ X(\delta_{\underline{n}}, w) \\
&= \delta_{jk}\ S_{i+M,l+M}(z)w^{-1}\ \delta(z/w)\ X(\delta_{\underline{m}}, z)\ X(\delta_{\underline{n}}, w) \\
&+\delta_{li}\ F(e_k, e_j)e_j(z)w^{-1}\ \delta(z/w)X(\delta_{\underline{m}}, z)\ X(\delta_{\underline{n}}, w) \\
&+\delta_{jk}\ \delta_{li}\ w^{-1}\ D_z\delta(z/w)X(\delta_{\underline{m}}, z)\ X(\delta_{\underline{n}}, w)\ (by\ S7\ ) \\
&= \delta_{jk}\ S_{i+M,l+M}(w)\ w^{-1}X(\delta_{\underline{m}} + \delta_{\underline{n}}, w)\ \delta(z/w) \\
&+\delta_{li}\ F(e_k, e_j)e_j\ (w)w^{-1}\ X(\delta_{\underline{m}} + \delta_{\underline{n}}, w)\ \delta(z/w) \\
&+\delta_{jk}\ \delta_{li}\ w^{-1}X(\delta_{\underline{m}+\underline{n}}, w)\ D_z\delta(z/w) \\
&-\delta_{jk}\ \delta_{li}\ w^{-1}\delta_{\underline{m}}(w)X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w)\ \ (By\ Lemma\ 4.7\ and\ 4.8\ )
\end{aligned}
$$

We now verify the 2nd part and 4th part of $SR7$ Consider for $k \neq j$

$$[S^{\underline{n}}_{k,l+M}(z), S^{\underline{m}}_{i+M,j}(w)] = [S_{k,l+M}(z)X(\delta_{\underline{n}}, z), S_{i+M,j}(w)X(\delta_{\underline{m}}, w)] \text{ (by 4.5)}$$
$$= [S_{k,l+M}(z), S_{i+M,j}(w)]\ X(\delta_{\underline{n}}, z)\ X(\delta_{\underline{m}}, w)$$
$$= \delta_{jk}\ S_{i+M,l+M}(z)w^{-1}\ \delta(z/w)\ X(\delta_{\underline{n}}, z)\ X(\delta_{\underline{m}}, w)$$
$$+\delta_{li}\ F(e_k, e_j)X(\alpha_{kj}, z)\ z^{-1}\ w^{-1}\ \delta(z/w)X(\delta_{\underline{n}}, z)\ X(\delta_{\underline{m}}, w)$$
$$-\delta_{jk}\ \delta_{li}\ w^{-1}\ D_z\delta(z/w)X(\delta_{\underline{n}}, z)\ X(\delta_{\underline{m}}, w) \text{ (by } S7 \text{)}$$
$$= \delta_{jk}\ S_{i+M,l+M}(w)\ w^{-1}X(\delta_{\underline{m}} + \delta_{\underline{n}}, w)\ \delta(z/w)$$
$$+\delta_{li}\ F(e_k, e_j)\ X(\alpha_{kj}, w)\ w^{-2}\ \delta(z/w)X(\delta_{\underline{m}} + \delta_{\underline{n}}, w)$$
$$-\delta_{jk}\ \delta_{li}\ X(\delta_{\underline{m}+\underline{n}}, w)\ w^{-1}\ D_z\delta(z/w)$$
$$+\delta_{jk}\ \delta_{li}\ w^{-1}\delta_{\underline{n}}(w)X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w))$$

for $k = j$

$$[S^{\underline{n}}_{k,l+M}(z), S^{\underline{m}}_{i+M,j}(w)] = [S_{k,l+M}(z)X(\delta_{\underline{n}}, z), S_{i+M,j}(w)X(\delta_{\underline{m}}, w)] \text{ (by 4.5)}$$
$$= [S_{k,l+M,j}(z), S_{i+M,j}(w)]\ X(\delta_{\underline{n}}, z)\ X(\delta_{\underline{m}}, w)$$
$$= \delta_{jk}\ S_{i+M,l+M}(z)w^{-1}\ \delta(z/w)\ X(\delta_{\underline{n}}, z)\ X(\delta_{\underline{m}}, w)$$
$$+\delta_{li}\ F(e_k, e_j)\ e_j(z)\ w^{-1}\ \delta(z/w)X(\delta_{\underline{n}}, z)\ X(\delta_{\underline{m}}, w)$$
$$-\delta_{jk}\ \delta_{li}\ w^{-1}\ D_z\delta(z/w)X(\delta_{\underline{n}}, z)\ X(\delta_{\underline{m}}, w) \text{ (by } S7 \text{)}$$
$$= \delta_{jk}\ S_{i+M,l+M}(z)\ w^{-1}X(\delta_{\underline{m}} + \delta_{\underline{n}}, w)\ \delta(z/w)$$
$$+\delta_{li}\ F(e_k, e_j)\ e_j(w)\ w^{-1}\ X(\delta_{\underline{m}} + \delta_{\underline{n}}, w)\ \delta(z/w)$$
$$-\delta_{jk}\ \delta_{li}\ w^{-1}\ X(\delta_{\underline{m}+\underline{n}}, w)\ D_z\delta(z/w)$$
$$+\delta_{jk}\ \delta_{li}\ w^{-1}\delta_{\underline{n}}(w)X(\delta_{\underline{m}+\underline{n}}, w)\ \delta(z/w))$$

Note that we are useing Lemmas 4.7 and 4.8 in the above calculations.

Remark 4.10

1. The central element $t^{\overline{m}}K_i, 1 \leq i \leq q-1$ acts as $T^{\delta_i}_{m_q}(\delta_{\underline{m}})$. The central element $t^{\overline{m}}K_q$ act as $X_{m_q}(\delta_{\underline{m}})$. In particular K_q acts as $X_0(0) = I_d$.

2. The module $V[\overline{\Gamma}] \otimes \mathfrak{F}$ is integrable with respect to $sl(\hat{M})$ but not integrable with respect to $sl(\hat{N})$ the second component of the even part. See $[EZ]$ and $[EF]$ for some interesting results on integrable modules.

3. The module $V[\overline{\Gamma}] \otimes \mathfrak{F}$ is not irreducible.
   For example $\delta_{\underline{m}}(k)(V[\overline{\Gamma}] \otimes \mathfrak{F})$ is always a proper sub module for any $\underline{m} \in \mathbb{Z}^{q-1}$ and any $k < 0$.

4. The structure of submodules and quotient modules of $V[\overline{\Gamma}] \otimes \mathfrak{F}$ will be investigated in a subsequent paper.

5. Theorem (4.5), when restricted to $sl(\hat{M})$, recover the main result of $[EM]$ and $[EMY]$. In fact the proof here as much simpler than $[EM]$. In the Lie algebra case our proof works for any simply laced case.

6. It is easy to see the representation is faithful. The only nontrivial thing is to prove that the full center acts faithfully. This follows from $[EM]$.

Acknowledgements: I thank Drazen Adamovic for some clarification on super vertex operator algebras.

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